Linear elliptic partial differential equations

The dual spaces $(W_0^{1,p}(\Omega))^*$, $(W^{1,p}(\Omega))^*$, $1 \le p < \infty$

If $f^0, f^1, \ldots, f^n \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\langle \tilde{f}, u \rangle := \int_\Omega f^0 u + \sum_{i=1}^n f^i \frac{\partial u}{\partial x_i}$$

defines a functional $\tilde{f} \in (W_0^{1,p}(\Omega))^*$ and $\hat{f} \in (W^{1,p}(\Omega))^*$.

As we will see now, every functional is of that form.

Theorem 1 If $\hat{f} \in (W^{1,p}(\Omega))^*$ (or $\tilde{f} \in (W_0^{1,p}(\Omega))^*$), $1 \le p < \infty$, then there are functions

$$f^0, f^1, \ldots, f^n \in L^{p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1$$

such that

$$\langle \tilde{f}, u \rangle = \int_\Omega f^0 u + \sum_{i=1}^n f^i \frac{\partial u}{\partial x_i}$$

for all $u \in W^{1,p}(\Omega)$ (or $u \in W_0^{1,p}(\Omega)$)

Proof We will prove the result in the case of $W^{1,p}(\Omega)$. Exactly the same proof works for $W_0^{1,p}(\Omega)$, one only needs to add subscript "o" whenever $W^{1,p}$ appears.

The mapping

$$W^{1,p}(\Omega) \ni u \xmapsto{T} (u, \nabla u) \in L^p(\Omega, \mathbb{R}^{n+1})$$

defines an isometric isomorphism between $W^{1,p}(\Omega)$ and a closed subspace $V \subset L^p(\Omega, \mathbb{R}^{n+1})$. Thus every functional $\tilde{f} \in (W^{1,p}(\Omega))^*$ defines a functional $T_{\tilde{f}} \in V^*$ by

$$\langle T_{\tilde{f}}, \omega \rangle = \langle \tilde{f}, T^{-1}\omega \rangle \quad \text{for } \omega \in V$$

i.e.,

$$\langle T_{\tilde{f}}, (u, \nabla u) \rangle = \langle \tilde{f}, u \rangle.$$

By the Hahn-Banach theorem, $T_{\tilde{f}}$ can be extended to a bounded functional in $(L^p(\Omega, \mathbb{R}^{n+1}))^* = L^{p'}(\Omega, \mathbb{R}^{n+1})$. That means, there are functions $f^0, f^1, \dots, f^n \in L^{p'}(\Omega)$ such that

$$\langle \tilde{f}, u \rangle = \langle T_{\tilde{f}}, (u, \nabla u) \rangle = \int_\Omega f^0 u + \sum_{i=1}^n f^i \frac{\partial u}{\partial x_i}. \quad \square$$

If $f \in L^1_{loc}(\Omega)$, then the <u>distributional derivative</u> $\frac{\partial f}{\partial x_i}$ is a functional on $C_0^\infty(\Omega)$ defined by

$$\left\langle \frac{\partial f}{\partial x_i}, \varphi \right\rangle := -\int_\Omega f \frac{\partial \varphi}{\partial x_i}.$$

The distributional derivative is more general than the weak derivative since it can be defined for <u>any</u> $f \in L^1_{loc}$. It is a weak derivative if the functional $\partial f / \partial x_i$ can be represented as

$$\left\langle \frac{\partial f}{\partial x_i}, \varphi \right\rangle = \int_\Omega g \varphi, \quad g \in L^1_{loc}(\Omega).$$

Then $\partial f / \partial x_i = g$.

Let $\tilde{f} \in (W_0^{1,p}(\Omega))^*$. If $\varphi \in C_0^\infty(\Omega) \subset W_0^{1,p}(\Omega)$, then

$$\langle \tilde{f}, \varphi \rangle = \int_\Omega f^0 \varphi + \sum_{i=1}^n f^i \frac{\partial \varphi}{\partial x_i} =$$

$$= \int_\Omega f^0 \varphi - \sum_{i=1}^n \langle \underbrace{\frac{\partial f^i}{\partial x_i}}_{\text{distrib. derivative}}, \varphi \rangle = \langle f^0 - \sum_{i=1}^n \frac{\partial f^i}{\partial x_i}, \varphi \rangle.$$

Hence $\tilde{f} \in (W_0^{1,p}(\Omega))^*$ restricted to $C_0^\infty(\Omega)$ equals

(1) $$\tilde{f} = f^0 - \sum_{i=1}^n \frac{\partial f^i}{\partial x_i} \quad \text{on} \quad C_0^\infty(\Omega).$$

Note that $\tilde{f} \in (W_0^{1,p}(\Omega))^*$ is uniquely determined by its restriction to $C_0^\infty(\Omega)$ because of density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$.

For this reason we can rewrite Theorem 1 by saying that functionals $\tilde{f} \in (W_0^{1,p}(\Omega))^*$ are of the form

(2) $$\hat{f} = f^0 - \sum_{i=1}^n \frac{\partial f^i}{\partial x_i}, \quad f^0, f^1, \ldots, f^i \in L^{p'}(\Omega)$$

Note that the right hand side of (2) is defined as a functional on $C_0^\infty(\Omega)$ only, but since $\tilde{f}$ restricted to $C_0^\infty(\Omega)$ uniquely determines

$\tilde{f}$ on $W_0^{1,p}(\Omega)$, writing (2) can be justified.

If $\tilde{f} \in (W^{1,p}(\Omega))^*$, then for the same reason as above, the restriction of $\tilde{f}$ to $C_0^\infty(\Omega)$ is of the form (1). However, since $C_0^\infty(\Omega)$ is not dense in $W^{1,p}(\Omega)$, the values of $\tilde{f}$ on $C_0^\infty(\Omega)$ do not determine $\tilde{f}$ on $W^{1,p}(\Omega)$ and hence we cannot say that functionals in $(W^{1,p}(\Omega))^*$ are of the form (2).

If $f \in W^{k,p}$, then $\frac{\partial f}{\partial x_i} \in W^{k-1,p}$

If $f \in W^{1,p}$, then $\frac{\partial f}{\partial x_i} \in L^p := W^{0,p}$.

Thus it is natural to adopt the convention that if $f \in L^p = W^{0,p}$, then $\frac{\partial f}{\partial x_i} \in W^{-1,p}$.

This convention explains commonly used notation

$$(W_0^{1,p}(\Omega))^* = W^{-1,p'}(\Omega)$$

to denote the dual space. If $p=2$ we write

$$(W_0^{1,2}(\Omega))^* = W^{-1,2}(\Omega)$$

or

$$(H_0^1(\Omega))^* = H^{-1}(\Omega).$$

The Dirichlet problem

Consider now the Dirichlet problem

$$(3) \quad \begin{cases} Lu = f & \text{in } \Omega \\ u \in W^{1,2}_{\omega}(\Omega) \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is bounded, $f \in L^2(\Omega)$ and $\omega \in W^{1,2}(\Omega)$.

We assume that L is uniformly elliptic in the following sense

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

$$a^{ij}, b^i, c \in L^\infty, \quad a^{ij} = a^{ji} \text{ a.e.}$$

$$(4) \quad \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \text{ for some } \theta > 0 \text{ and almost all } x \in \Omega$$

Condition (3) is called uniform ellipticity of L. It means that the symmetric matrix $[a^{ij}]$ is positive definite.

We say that u is a weak solution to (3) if

$$(5) \quad \int_\Omega \sum_{i,j} a^{ij}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} + \sum_i b^i \frac{\partial u}{\partial x_i}\varphi + cu\varphi = \int_\Omega f\varphi$$

for all $\varphi \in C_0^\infty(\Omega)$ and hence, by density of $C_0^\infty(\Omega)$ in $W_0^{1,2}(\Omega)$, for all $\varphi \in W_0^{1,2}(\Omega)$

We will show now that the problem (3) can be reduced to an equivalent problem, but with zero boundary data.

If $u \in W^{1,2}_{\omega}(\Omega)$ is a solution to $Lu = f$, then $\hat{u} = u - \omega \in W_0^{1,2}(\Omega)$ and

$$L\hat{u} = Lu - L\omega = f - L\omega =$$

$$\underbrace{f - c\omega - \sum_i b^i \frac{\partial \omega}{\partial x_i}}_{f^0 \in L^2} - \sum_j \frac{\partial}{\partial x_j}\underbrace{\left(-\sum_i a^{ij}\frac{\partial \omega}{\partial x_i}\right)}_{f^j \in L^2}$$

so the Dirichlet problem (3) is equivalent to

$$(6)\quad \begin{cases} L\hat{u} = \tilde{f} = f^0 - \sum_j \frac{\partial f^j}{\partial x_j} \in W^{-1,2}(\Omega) \\ \hat{u} \in W_0^{1,2}(\Omega) \end{cases}$$

If, for example, $\omega \in W^{2,2}(\Omega)$ and $a^{ij} \in C^1(\bar{\Omega})$, then actually $\tilde{f} \in L^2(\Omega)$. Since (3) can be reduced to (6), we will only study the problems

$$(7)\quad \begin{cases} Lu = f \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

where $f \in L^2(\Omega)$ or $f \in W^{-1,2}(\Omega) = (W_0^{1,2}(\Omega))^*$.

Thus $u \in W_0^{1,2}(\Omega)$ is a weak solution to (7) if

(8) $$\int_\Omega \sum_{ij} a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \sum_i b^i \frac{\partial u}{\partial x_i} \varphi + cu\varphi = \langle f, \varphi \rangle$$

for all $\varphi \in W_0^{1,2}(\Omega)$. Here $\langle f, \varphi \rangle = \int_\Omega f\varphi$ if $f \in L^2(\Omega)$ and, more generally $\langle f, \varphi \rangle$ is the evaluation of the functional $f \in W^{-1,2}(\Omega)$ on $\varphi \in W^{1,2}(\Omega)$.

The bilinear form associated with L is defined by

$$B : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \longrightarrow \mathbb{R}$$

$$B[u,v] = \int_\Omega \sum_{ij} a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_i b^i \frac{\partial u}{\partial x_i} v + cuv$$

so $u \in W_0^{1,2}(\Omega)$ is a weak solution of (7) if

$B[u,v] = \langle f, v \rangle$ for all $v \in W_0^{1,2}(\Omega)$,

see (8).

The existence of a unique solution of (7), under some additional assumptions, will be concluded from

Theorem 2 (Lax-Milgram) Assume that

$$\mathcal{B} : H \times H \longrightarrow \mathbb{R}$$

is a bilinear form on a real Hilbert space H such that for some constants $\alpha, \beta > 0$, we have

$$|B[u,v]| \leq \alpha \|u\| \|v\|, \quad u, v \in H$$

and

$$B[u,u] \geq \beta \|u\|^2, \quad u \in H.$$

Then for any bounded linear functional $f \in H^*$ there is a unique $u \in H$ such that

$$B[u,v] = \langle f, v \rangle \quad \text{for all } v \in H.$$

Proof

① For every $u \in H$, $B[u, \cdot] \in H^*$ so there is a unique $w \in H$ such that $B[u, \cdot] = \langle w, \cdot \rangle$.
We will write $w = Au$ so

$$\forall v \quad B[u,v] = \langle Au, v \rangle.$$

It is easy to see that $A : H \to H$ is bounded and linear. Linearity is obvious and boundedness follows from

$$\|Au\|^2 = \langle Au, Au \rangle = B[u, Au] \leq \alpha \|u\| \|Au\|$$

$$\|Au\| \leq \alpha \|u\|.$$

② $A : H \to H$ is injective and $R(A)$ $(= \operatorname{Im} A)$ is a closed subspace of H.

We have

$$\beta \|u\|^2 \le B[u,u] = \langle Au, u\rangle \le \|Au\|\,\|u\|$$

$$\beta\|u\| \le \|Au\|.$$

This implies that A is injective. It also implies that the image of A is a closed subspace of H. To prove closedness we need to show that

$$Au_k \to w \implies w = Au \quad \text{for some } u.$$

We have

$$\beta\|u_k - u_\ell\| \le \|Au_k - Au_\ell\| \xrightarrow{k,\ell\to\infty} 0$$

so (u_k) is a Cauchy sequence and hence it is convergent, $u_k \to u$. Hence

$$Au = \lim_k Au_k = w.$$

③ $R(A) = H$ so $A : H \to H$ is an isomorphism. Suppose to the contrary that $0 \ne w \in R(A)^\perp$ (we use here the fact that $R(A)$ is closed). Then

$$\beta\|w\|^2 \le B[w,w] = \langle Aw, w\rangle = 0$$

(the last equality is a consequence of $Aw \in R(A)$, $w \in R(A)^\perp$) so $w = 0$ which is a contradiction.

④ Now we can complete the proof. If $f \in H^*$, then there is $w \in H$ such that $\langle f, \cdot\rangle = \langle w, \cdot\rangle$. Since $A : H \to H$

is an isomorphism, there is a unique u such that $Au = \omega$ and hence

$B[u,\cdot] = \langle Au,\cdot\rangle = \langle \omega,\cdot\rangle = \langle f,\cdot\rangle$.

Uniqueness of u satisfying $B[u,\cdot] = \langle f,\cdot\rangle$ is easy to check. $\square$

Recall that the bilinear form associated with the uniformly elliptic operator L is

$$B[u,v] = \int_\Omega \sum_{ij} a^{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_i b^i \frac{\partial u}{\partial x_i} v + cuv,$$

$$a^{ij}, b^i, c \in L^\infty, \quad a^{ij} = a^{ji}, \quad \sum_{ij} a^{ij}\xi_i\xi_j \geq \theta|\xi|^2.$$

<u>Theorem 3 (Energy estimates)</u> Let B be the bilinear form associated with an uniformly elliptic operator L. Then there are constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

$$|B[u,v]| \leq \alpha \|u\|_{1,2}\|v\|_{1,2}$$

$$B[u,u] + \gamma\|u\|_2^2 \geq \beta\|u\|_{1,2}^2$$

for all $u, v \in W_0^{1,2}(\Omega)$.

<u>Proof</u> The first inequality is easy

$$|B[u,v]| \leq \sum_{ij}\|a^{ij}\|_\infty \int_\Omega |\nabla u||\nabla v| + \sum_i \|b^i\|_\infty \int_\Omega |\nabla u||v| + \|c\|_\infty \int_\Omega |u||v| \leq \alpha\|u\|_{1,2}\|v\|_{1,2}.$$

In the proof of the second estimate we need to use ellipticity of L

$$\theta \int_\Omega |\nabla u|^2 \leq \int_\Omega \sum_{ij} a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

$$= B[u,u] - \int_\Omega \left(\sum_i b^i \frac{\partial u}{\partial x_i} u + c|u|^2 \right) dx$$

$$\leq B[u,u] + \sum_i \|b^i\|_\infty \int_\Omega |\nabla u||u| + \|c\|_\infty \int_\Omega |u|^2$$

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$$

$$\int_\Omega |\nabla u||u| \leq \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon} \int_\Omega |u|^2.$$

Taking $\varepsilon > 0$ so small that

$$\varepsilon \sum_i \|b^i\|_\infty \leq \frac{\theta}{2}$$

yields

$$\frac{\theta}{2} \int_\Omega |\nabla u|^2 \leq B[u,u] + \gamma \int_\Omega |u|^2 \quad \text{for some } \gamma > 0$$

Since by the Poincaré inequality (recall that Ω is bounded)

$$\int_\Omega |\nabla u|^2 \approx \|u\|_{1,2}^2 \quad \text{for } u \in W_0^{1,2}(\Omega)$$

we conclude that

$$\beta \|u\|_{1,2}^2 \leq B[u,u] + \gamma \|u\|_2^2, \quad u \in W_0^{1,2}(\Omega). \quad \square$$

Theorem 4 (First Existence Theorem) There is $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each $f \in L^2(\Omega)$, the boundary value problem

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

has a unique solution.

Proof Let μ be as in Theorem 3. For $\mu \geq \gamma$, $L_\mu = Lu + \mu u$ is an elliptic operator with the bilinear form

$$B_\mu[u,v] = B[u,v] + \mu \int_\Omega uv = B[u,v] + \mu \langle u,v \rangle.$$

Clearly

$$|B_\mu[u,v]| \leq (\alpha + \mu) \|u\|_{1,2} \|v\|_{1,2}$$

and

$$B_\mu[u,u] = B[u,u] + \mu \|u\|_2^2 \geq B[u,u] + \gamma \|u\|_2^2 \geq \beta \|u\|_{1,2}^2$$

Thus we can apply the Lax-Milgram theorem to the bilinear form B_μ.

Since $f \in L^2(\Omega)$ defines an element in $(W_0^{1,2}(\Omega))^*$ through the standard scalar product $v \mapsto \int_\Omega fv$, according to the Lax-Milgram theorem there is a unique $u \in W_0^{1,2}(\Omega)$ such that

$$B_\mu[u,v] = \langle f, v \rangle \text{ for all } v \in W_0^{1,2}(\Omega).$$

In other words, u is a weak solution to

$$Lu + \mu u = f, \quad u \in W_0^{1,2}(\Omega). \quad \square$$

Fredholm alternative

Assume that the coefficients of L are sufficiently smooth. We want to find the formal adjoint operator L^* such that

$$\forall u, v \in C_0^\infty(\Omega) \quad \langle Lu, v\rangle = \langle u, L^* v\rangle$$

Since $B[u,v] = \langle Lu, v\rangle$, the adjoint bilinear form will be

$$B^*[v,u] = \langle L^* v, u\rangle = \langle Lu, v\rangle = B[u,v].$$

We have

$$\langle Lu, v\rangle = \int_\Omega \left(-\sum_{ij} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) + \sum_i b^i \frac{\partial u}{\partial x_i} + cu\right) v$$

$$= \int_\Omega \sum_{ij} a^{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} - u \sum_i \frac{\partial}{\partial x_i}(b^i v) + c v u$$

$$= \int_\Omega -\sum_{ij} \frac{\partial}{\partial x_i}\left(a^{ij}\frac{\partial v}{\partial x_j}\right) u - \sum_i b^i \frac{\partial v}{\partial x_i} u + \left(c - \sum_i \frac{\partial b^i}{\partial x_i}\right) v u$$

$$= \langle u, L^* v\rangle$$

where

$$(9) \quad L^* v = -\sum_{ij} \frac{\partial}{\partial x_i}\left(a^{ij}\frac{\partial v}{\partial x_j}\right) - \sum_i b^i \frac{\partial v}{\partial x_i} + \left(c - \sum_i \frac{\partial b^i}{\partial x_i}\right)$$

Note that we need b^i to be of class C^1 as otherwise the derivatives $\partial b^i / \partial x_i$ would not be defined.

<u>Definition</u> Let L be an uniformly elliptic operator and assume in addition that $b^i \in C^1(\bar{\Omega})$, $i = 1, 2, \dots, n$. Then the formal adjoint operator L^* is defined by (9) and the adjoint bilinear form

$$B^* : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$$

by
$$B^*[v, u] = B[u, v]$$

Thus $v \in W_0^{1,2}(\Omega)$ is a weak solution to

$$\begin{cases} L^* v = f & \text{in } \Omega \\ v \in W_0^{1,2}(\Omega) \end{cases}$$

if
$$B^*[v, u] = \langle f, u \rangle \text{ for all } u \in W_0^{1,2}(\Omega).$$

<u>Theorem 5 (Fredholm alternative I)</u>

Let L be an uniformly elliptic operator. Then the following conditions are equivalent

(α) For every $f \in L^2(\Omega)$ there is a unique solution of
$$\begin{cases} Lu = f \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

(β) $u = 0$ is the only solution of
$$\begin{cases} Lu = 0 \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

Implication $(\alpha) \Rightarrow (\beta)$ is obvious, but the other one $(\beta) \Rightarrow (\alpha)$ is not. It is only obvious that if $Lu = f$ has a solution for some $f \in L^2(\Omega)$, then the solution is unique. There is no apparent reason why for $0 \neq f \in L^2(\Omega)$ the equation $Lu = f$ should have a solution.

Theorem 6 (Fredholm Alternative II)

Let L be an uniformly elliptic operator and $b^i \in C^1(\bar{\Omega})$ (so we can define L^*). Then the spaces N and N^* of solutions of the Dirichlet problems

$$\begin{cases} Lu = 0 \\ u \in W_0^{1,2}(\Omega) \end{cases} \quad \text{and} \quad \begin{cases} L^* v = 0 \\ v \in W_0^{1,2}(\Omega) \end{cases}$$

respectively, have finite and equal dimensions

$$\dim N = \dim N^* < \infty.$$

Moreover, given $f \in L^2(\Omega)$, the Dirichlet problem

$$\begin{cases} Lu = f \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

has a solution if and only if $f \perp N^*$ i.e.

$$\langle f, v \rangle = 0 \quad \text{for all } v \in N^*.$$

Observe that Theorem 5 follows from Theorem 6 provided we assume in Theorem 5 that $b^i \in C^1(\Omega)$ (so we can apply Theorem 6). Since the implication $(\alpha) \Rightarrow (\beta)$ is obvious, it remains to prove that $(\beta) \Rightarrow (\alpha)$. Condition (β) means that $N = \{0\}$. Hence $\dim N^* = \dim N = 0$, $N^* = \{0\}$. Thus every $f \in L^2(\Omega)$ is orthogonal to N^*, $f \perp N^*$ and hence for every $f \in L^2(\Omega)$ the equation $Lu = f$ has a solution by Theorem 6. Uniqueness follows from (β).

Will prove both Theorems 5 and 6 at the same time. Since in Theorem 5 we do not assume $b^i \in C^1(\Omega)$ we cannot conclude it from Theorem 6.

<u>Proof of Theorems 5 and 6</u> Let L be an uniformly elliptic operator. We do not assume yet that $b^i \in C^1(\bar{\Omega})$. Let $\gamma \geq 0$ be as in Theorem 4 so for every $g \in L^2(\Omega)$, there is a unique solution to

$$L_\gamma u := Lu + \gamma u = g, \quad u \in W_0^{1,2}(\Omega).$$

That is

$$B_\gamma[u,v] := B[u,v] + \gamma\langle u,v\rangle = \langle g, v\rangle$$

for every $v \in W_0^{1,2}(\Omega)$. Denote this solution by

$$u := L_\gamma^{-1} g.$$

We claim that the operator

$$L^2(\Omega) \ni g \longmapsto Kg := \gamma L_\gamma^{-1} g \in W_0^{1,2}(\Omega) \subset L^2(\Omega)$$

$$K : L^2(\Omega) \longrightarrow L^2(\Omega)$$

is compact.

Note that Kg solves the equation $L_\gamma u = \gamma g$ i.e.

$$B_\gamma[Kg, v] = \langle \gamma g, v\rangle, \quad v \in W_0^{1,2}(\Omega)$$

so the energy estimates from Theorem 4 yield

$$\beta \|Kg\|_{1,2}^2 \leq B_\gamma[Kg, Kg] = \langle \gamma g, Kg\rangle$$

$$\leq \gamma \|g\|_2 \|Kg\|_2 \leq \gamma \|g\|_2 \|Kg\|_{1,2}$$

$$\|Kg\|_{1,2} \leq \frac{\gamma}{\beta} \|g\|_2$$

Hence

$$K : L^2(\Omega) \longrightarrow W_0^{1,2}(\Omega)$$

is bounded. Since the embedding $W_0^{1,2}(\Omega) \subset\subset L^2(\Omega)$ is compact, the operator

$$L^2(\Omega) \xrightarrow[\text{bounded}]{K} W_0^{1,2}(\Omega) \underset{\text{compact}}{\subset\subset} L^2(\Omega)$$

$$K : L^2(\Omega) \longrightarrow L^2(\Omega)$$

is compact.

Now u is a solution to

$$(10) \qquad \begin{cases} Lu = f \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

iff

$$L_\gamma u = Lu + \gamma u = \gamma u + f$$

iff

$$\begin{aligned} u &= L_\gamma^{-1}(\gamma u + f) \\ &= \gamma L_\gamma^{-1} u + \underbrace{L_\gamma^{-1} f}_{h} \\ &= Ku + h. \end{aligned}$$

That is u is a solution to (10) iff

$$u - Ku = h, \text{ where } h = L_\gamma^{-1} f.$$

To complete the proof we need to use some facts about compact operators in Hilbert spaces. Recall that if $A : H \longrightarrow H$ is a bounded operator, then we can define the adjoint operator $A^* : H \longrightarrow H$ such that $\langle Au, v \rangle = \langle u, A^* v \rangle$ for all $u, v \in H$.

We will need the following results that we state without proof.

Theorem 7. $K: H \to H$ is compact if and only if $K^*: H \to H$ is compact.

For a bounded operator A by $N(A)$ and $R(A)$ we denote the kernel and the range respectively.

Theorem 8 (Fredholm alternative)

Let $K: H \to H$ be compact. Then

(1) $\dim N(I-K) = \dim N(I-K^*) < \infty$.

(2) $R(I-K)$ is closed.

(3) $R(I-K) = N(I-K^*)^{\perp}$

(4) $N(I-K) = \{0\}$ iff $R(I-K) = H$.

Although we will not prove this result we will prove that (3) $\Rightarrow$ (2) and (1) & (3) $\Rightarrow$ (4).

Since the orthogonal complement of any subspace is closed, (3) implies (2).

Also (1) and (3) imply (4). Indeed, by (1) $N(I-K) = \{0\}$ iff $N(I-K^*) = \{0\}$ which is equivalent to

$$R(I-K) \underset{\text{by } (3)}{=} N(I-K^*)^{\perp} = H.$$

This proves (4).

In order to prove Theorem 5 it suffices to show that $(\beta) \Rightarrow (\alpha)$ since the implication $(\alpha) \Rightarrow (\beta)$ is obvious.

Recall that u is a solution to

$$(11) \qquad Lu = f, \quad u \in W_0^{1,2}(\Omega)$$

if and only if $u - Ku = h$, $h = L_\gamma^{-1} f$.

If (β) is satisfied, then the only solution to $Lu = 0$ is $u = 0$ so the only solution to $u - Ku = 0$ is $u = 0$, that is $N(I-K) = 0$ which, according to Theorem 8 (4) implies that $R(I-K) = L^2(\Omega)$.

That is for $h = L_\gamma^{-1} f$ there is $u \in L^2(\Omega)$ such that $u - Ku = h$.

Clearly $u = Ku + L_\gamma^{-1} f \in W_0^{1,2}(\Omega)$ and so u is a solution to (11). This proves existence of a solution in (α) for every $f \in L^2(\Omega)$. Uniqueness follows from (β).

Now we will complete the proof of Theorem 6.

We observed that u is a solution to $Lu = 0$, $u \in W_0^{1,2}$ iff $u \in N(I-K)$ i.e. $N = N(I-K)$. It is easy to see that $N^* = N(I-K^*)$. Hence by theorem 8(1)

$$\dim N = \dim N^* < \infty.$$

This completes the proof of the first part of theorem 6.

If $Lu = f \in L^2$ has a solution u, then $u - Ku = h$, $h = L_\gamma^{-1} f = \frac{1}{\gamma} K f$ so $h \in R(I-K) = N(I-K^*)^\perp = (N^*)^\perp$.

that is

$$\langle h, v \rangle = 0 \quad \text{for all } v \in N^*$$

i.e. for all v such that $v = K^* v$.

Thus for every $v \in N^*$

$$\frac{1}{\gamma} \langle f, v \rangle = \frac{1}{\gamma} \langle f, K^* v \rangle = \frac{1}{\gamma} \langle Kf, v \rangle$$
$$= \langle h, v \rangle = 0.$$

Conversly, if for every $v \in N^*$, $\langle f, v \rangle = 0$, then

$$0 = \frac{1}{\gamma} \langle f, v \rangle = \frac{1}{\gamma} \langle f, K^* v \rangle = \frac{1}{\gamma} \langle Kf, v \rangle$$
$$= \langle h, v \rangle.$$

so $h = L_\gamma^{-1} f \in (N^*)^\perp = R(I-K)$
and hence the equation

$$u - Ku = h, \quad h = L_\gamma^{-1} f$$

has a solution which is a solution to
$Lu = f$. $\square$

The spectrum of L

If γ is as in Theorem 4, then for $\mu \geq \gamma$

$$(12) \qquad \begin{cases} Lu + \mu u = f \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

has a unique solution. If we rewrite this equation as

$$(13) \qquad \begin{cases} Lu = \lambda u + f, & \lambda = -\mu \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

then this equation has a unique solution for $\lambda \leq -\gamma$. However, Theorem 4 does not say anything about solutions to (13) when $\lambda > -\gamma$.
This case is answered by the next result.

<u>Theorem 9</u> There is a countable (or finite) set $\Sigma \subset \mathbb{R}$ such that

$$\begin{cases} Lu = \lambda u + f \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

has a unique solution for every $f \in L^2(\Omega)$ if and only if $\lambda \notin \Sigma$. If Σ is infinite, its elements can be arranged as an increasing sequence divergent to ∞,

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots \quad , \quad \lambda_k \to \infty .$$

The set Σ is called the <u>spectrum of the operator L</u>.

If $\lambda \in \Sigma$, then it is not true that $Lu = \lambda u + f$ has a unique solution for every f. Hence, according to Fredholm alternative (Theorem 5), $Lu = \lambda u$ has a non-trivial solution $0 \neq u \in W_0^{1,2}(\Omega)$. That is λ is an eigenvalue of L and u is an eigenfunction. Since $\dim N < \infty$, the eigenspace corresponding to a fixed eigenvalue has finite dimension.

In the proof we will need some results from the spectral theory of compact operators.

<u>Definition</u> Let $A: X \to X$ be a bounded operator in a Banach space X. The <u>resolvent set</u> of A is

$$\rho(A) = \{\lambda \in \mathbb{R} \mid A - \lambda I \text{ is an isomorphism}\}$$

<u>The spectrum of A</u> is

$$\sigma(A) = \mathbb{R} \setminus \rho(A) = \{\lambda \in \mathbb{R} \mid A - \lambda I \text{ is not an isomorphism}\}$$

If $\lambda \in \sigma(A)$ then we have two possibilities

$$N(A-\lambda I) \neq \{0\} \quad \text{or} \quad R(A-\lambda I) \neq H.$$

In the first case λ is an eigenvalue so clearly the <u>pointwise spectrum of A</u>

$$\sigma_p(A) = \{\lambda \in \mathbb{R} \mid \lambda \text{ is an eigenvalue of } A\}$$

is a subset of the spectrum, $\sigma_p(A) \subset \sigma(A)$.

In general it may happen that $\sigma(A) \setminus \sigma_p(A) \neq \phi$.

If $\lambda \in \sigma(A) \setminus \sigma_p(A)$, then the operator $A-\lambda I$ is injective, but not surjective.

The situation is much more elegant if A is a compact operator in a Hilbert space.

Theorem 10 Let $K: H \to H$ be compact, $\dim H = \infty$. Then

(i) $0 \in \sigma(K)$

(ii) $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$

(iii) $\sigma(K)$ is finite or it forms a sequence convergent to 0.

Compact operators are never onto ($\dim H = \infty$) so $0 \in \sigma(K)$, but it may happen that $0 \notin \sigma_p(K)$.

If $\lambda \neq 0$, then

$$N(K-\lambda I) = \{0\} \equiv R(K-\lambda I) = H$$

(Theorem 8 (4)) so

$$N(K-\lambda I) \neq \{0\} \equiv R(K-\lambda I) \neq H$$

and hence

$$\lambda \in \sigma(K) \implies \lambda \in \sigma_p(K).$$

This completes the proof of (i) and (ii). We will not prove (iii).

Proof of Theorem 9

If γ is as in Theorem 4, then for $\lambda \leq -\gamma$

$$(14) \qquad \begin{cases} Lu = \lambda u + f \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

has a unique solution for every $f \in L^2(\Omega)$. Note that we can assume in Theorem 4 that $\gamma > 0$. This shows that Σ is bounded from below by $-\gamma$. Thus assume that $\lambda > -\gamma$.

By Theorem 6, (14) has a unique solution for each $f \in L^2$ iff

$$\begin{cases} Lu = \lambda u \\ u \in W_0^{1,2}(\Omega) \end{cases} \Longrightarrow u = 0,$$

which is equivalent to

$$(15) \qquad \begin{cases} L_\gamma u := Lu + \gamma u = (\lambda+\gamma) u \\ u \in W_0^{1,2}(\Omega) \end{cases} \Longrightarrow u = 0.$$

Recall that for $f \in L^2$, $L_\gamma v = f$ has a unique solution $v = L_\gamma^{-1} f$ and the operator

$$K = \gamma L_\gamma^{-1} : L^2(\Omega) \to L^2(\Omega) \quad \text{is compact}.$$

Hence u is a solution to $L_\gamma u = (\lambda+\gamma) u$ iff $u = L_\gamma^{-1}((\lambda+\gamma)u) = \frac{\lambda+\gamma}{\gamma} K u$ so (15) is equivalent to

$$u = \frac{\lambda+\gamma}{\gamma} K u \Longrightarrow u = 0,$$

i.e.

$$\frac{\gamma}{\lambda+\gamma} \notin \sigma_p(K).$$

We proved that (14) has a unique solution for every $f \in L^2$ iff $\frac{\gamma}{\lambda+\gamma} \notin \sigma_p(K)$ so

$$\lambda \in \Sigma \quad \equiv \quad \frac{\gamma}{\lambda+\gamma} \in \sigma_p(K).$$

$\sigma_p(K)$ is finite or countable so is Σ.
If $\sigma_p(K)$ is infinite, then it forms a sequence convergent to 0

$$\frac{\gamma}{\lambda_k+\gamma} \longrightarrow 0 \qquad \text{i.e. } \lambda_k \longrightarrow \infty$$

and we can arrange the elements of Σ as an increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots, \qquad \lambda_k \to \infty.$$

The proof is complete. □

Spectral theorem

Assume now that the lower order terms of a uniformly elliptic operator L are zero so L is of the form

$$Lu = -\sum_{ij} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right), \quad a^{ij} = a^{ji}, \quad \sum_{ij} a^{ij}\xi_i\xi_j \geq \theta|\xi|^2.$$

- The corresponding bilinear form is symmetric and positive definite

$$B[u,v] = \int_\Omega \sum_{ij} a^{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} = B[v,u]$$

$$B[u,u] = \int_\Omega \sum_{ij} a^{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \geq \theta\int_\Omega |\nabla u|^2 \geq \beta\|u\|_{1,2}^2$$

- Eigenvalues of L are positive and greater than β. Indeed, if $Lu = \lambda u$, then
$$B[u, v] = \langle \lambda u, v \rangle, \quad v \in W_0^{1,2}(\Omega)$$
so
$$\beta \|u\|_2^2 < \beta \|u\|_{1,2}^2 \le B[u,u] = \langle \lambda u, u \rangle = \lambda \|u\|_2^2.$$

- Eigenvectors corresponding to different eigenvalues are orthogonal. Indeed, if $Lu_1 = \lambda_1 u_1$, $Lu_2 = \lambda_2 u_2$, $\lambda_1 \neq \lambda_2$, then
$$B[u_1, u_2] = \langle \lambda_1 u_1, u_2 \rangle,$$
$$B[u_2, u_1] = \langle \lambda_2 u_2, u_1 \rangle.$$
Since the left-hand sides are equal,
$$\langle \lambda_1 u_1, u_2 \rangle = \langle \lambda_2 u_2, u_1 \rangle$$
$$(\lambda_1 - \lambda_2) \langle u_1, u_2 \rangle = 0$$
$$\langle u_1, u_2 \rangle = 0.$$

- If λ is an eigenvalue of L, then the corresponding eigenspace E_λ has finite dimension (see p. 23), say $\dim E_\lambda = k$. Thus we can an orthonormal basis of E_λ (in L^2 norm), $u_1, \ldots, u_k$ so that $Lu_i = \lambda_k u_i$. We say that λ has multiplicity k.

Theorem 11 Let L be as above. Then repeating each eigenvalue according to its multiplicity, they will form a sequence
$$0<\lambda_1\leq\lambda_2\leq\lambda_3\leq\ldots,\quad \lambda_k\to\infty.$$
Moreover, there is an orthonormal basis $\{u_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$ consisting of eigenfunctions of L, $Lu_k=\lambda_k u_k$, $u_k\in W_0^{1,2}(\Omega)$.

Remark Later we will prove that if $a^{ij}\in C^\infty$ then $u_k\in C^\infty$ and if $a^{ij}\in C^\infty(\bar\Omega)$ and the boundary $\partial\Omega$ is smooth, then $u_k\in C^\infty(\bar\Omega)$.

Remark We already proved that eigenvalues are positive so they can be arranged as a sequence
$$0<\lambda_1\leq\lambda_2\leq\ldots$$
but we do not know yet that the sequence is infinite. We also proved that we can find an orthonormal basis of eigenfunctions of L of the direct sum of orthogonal eigenspaces

$$(16)\qquad \bigoplus_k E_{\lambda_k}$$

but we do not know yet that (16) equals $L^2(\Omega)$.

<u>Proof</u> Since $B[u,u] \geqslant \beta \|u\|_{1,2}^2$, the form B satisfies the assumptions of the Lax-Milgram theorem and hence for any $f \in L^2(\Omega)$ we can find a unique solution $u \in W_0^{1,2}(\Omega)$ of $Lu = f$. Denoting this solution by $u = L^{-1}f$, we obtain a compact operator

$$S := L^{-1} : L^2(\Omega) \longrightarrow L^2(\Omega).$$

We claim that the operator S is symmetric i.e.,

$$\langle Sf, g \rangle = \langle f, Sg \rangle, \quad f, g \in L^2(\Omega).$$

If $Sf = u$, then $Lu = f$ so

$$B[u,v] = \langle f, v \rangle \text{ for all } v \in W_0^{1,2}$$

If $Sg = v$, then $Lv = g$ so

$$B[v,u] = \langle g, u \rangle.$$

Hence

$$\langle f, Sg \rangle = \langle f, v \rangle = B[u,v] = B[v,u]$$
$$= \langle g, u \rangle = \langle g, Sf \rangle.$$

The result follows now from the spectral theorem for compact symmetric operators.

<u>Theorem 12</u> Let $S : H \to H$ be a compact and symmetric ($\langle Su, v \rangle = \langle u, Sv \rangle$ for all $u, v \in H$) operator on a separable Hilbert space H. Then there is a countable orthonormal basis of H consisting of eigenvectors of H.

There is an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $S = L^{-1}$.
If $u \neq 0$ is an eigenfunction of S, $Su = \eta u$, then $L(\eta u) = u$ so $\eta \neq 0$ and hence $Lu = \frac{1}{\eta} u$ so u is an eigenfunction of L.
Thus there is an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of L.
Since the eigenspaces of L are finitely dimensional, we must have infinitely many eigenvalues of L and the theorem follows.

Boundedness of solutions

Theorem 13 Let L be a uniformly elliptic operator as on p. 5. If $\lambda \notin \Sigma$, then the Dirichlet problem

$$\begin{cases} Lu = \lambda u + f \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

has a unique solution for every $f \in L^2(\Omega)$. In fact there is a constant

$$C = C(\lambda, \Omega, L) > 0$$

such that

$$\|u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

Proof Suppose to the contrary that we have a sequence of solutions

$$\begin{cases} Lu_k = \lambda u_k + f_k \\ u_k \in W_0^{1,2}(\Omega) \end{cases}$$

$$\|u_k\|_2 > k\|f_k\|_2$$

We can assume that $\|u_k\|_2 = 1$ (why?) so $f_k \to 0$ in $L^2(\Omega)$.

It follows from the energy estimates that the sequence $\{u_k\}$ is bounded in L^2.

Indeed,

$$B[u_k, v] = \langle \lambda u_k, v \rangle + \langle f_k, v \rangle,$$

$$B[u_k, u_k] = \lambda \langle u_k, u_k \rangle + \langle f_k, u_k \rangle$$
$$\leq \lambda + \|f_k\|_2,$$

$$\beta^2 \|u_k\|_{1,2}^2 \leq B[u_k, u_k] + \gamma \|u_k\|_2^2$$
$$\leq \lambda + \|f_k\|_2 + \gamma.$$

Using reflexivity of $W_0^{1,2}(\Omega)$ and compactness of the embedding $W_0^{1,2}(\Omega) \subset L^2(\Omega)$ we can find a subsequence $\{u_{k_j}\}$ so

$$u_{k_j} \rightharpoonup u \quad \text{in } W_0^{1,2}$$

$$u_{k_j} \longrightarrow u \quad \text{in } L^2 \text{ and hence } \|u\|_2 = 1.$$

Passing to the limit in

$$B[u_{k_j}, v] = \langle \lambda u_{k_j}, v \rangle + \langle f_{k_j}, v \rangle$$

yields

$$B[u, v] = \langle \lambda u, v \rangle$$

so

$$\begin{cases} Lu = \lambda u, \quad \|u\|_2 = 1, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

which contradicts the fact that λ is not an eigenvalue. $\square$