Schauder fixed point theorem

Let X be a real Banach space.

Theorem (Schauder)

Suppose $K \subset X$ is compact, convex and
$$A : K \longrightarrow K$$
is continuous. Then A has a fixed point in K i.e, there is $u \in K$ such that $A(u) = u$.

Proof Fix $\varepsilon > 0$ and choose $u_1, \dots, u_{N_\varepsilon} \in K$ such that
$$K \subset \bigcup_{i=1}^{N_\varepsilon} B(u_i, \varepsilon).$$

Existence of such a finite covering follows from compactness of K. Choosing ε such that $2\varepsilon < \operatorname{diam} K$ we have that
$K \setminus B(u_i, \varepsilon) \neq \emptyset$ for all $i = 1, 2, \dots, N_\varepsilon$.
Let K_ε be the convex hull of the set $\{u_1, \dots, u_{N_\varepsilon}\}$ i.e,
$$K_\varepsilon = \left\{ \sum_{i=1}^{N_\varepsilon} \lambda_i u_i \;\middle|\; 0 \le \lambda_i \le 1, \ \sum_{i=1}^{N_\varepsilon} \lambda_i = 1 \right\}.$$
Then $K_\varepsilon \subset K$ since K is convex.

Define $P_\varepsilon : K \to K_\varepsilon$ by

$$P_\varepsilon(u) = \frac{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}(u, K \setminus B(u_i, \varepsilon))\, u_i}{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}(u, K \setminus B(u_i, \varepsilon))}$$

The denominator is always $\neq 0$ because each $u \in K$ belongs to some $B(u_i, \varepsilon)$ and hence $\operatorname{dist}(u, K \setminus B(u_i, \varepsilon)) > 0$. Also $P_\varepsilon(u) \in K_\varepsilon$ because $P_\varepsilon(u) = \sum_i \lambda_i u_i$ where

$$\lambda_i = \frac{\operatorname{dist}(u, K \setminus B(u_i, \varepsilon))}{\sum_j \operatorname{dist}(u, K \setminus B(u_j, \varepsilon))}.$$

Clearly P_ε is continuous and for every $u \in K$

$$\| P_\varepsilon(u) - u \| \leqslant \frac{\sum_i \operatorname{dist}(u, K \setminus B(u_i, \varepsilon)) \|u_i - u\|}{\sum_i \operatorname{dist}(u, K \setminus B(u_i, \varepsilon))} \leqslant \varepsilon$$

because

$$\operatorname{dist}(u, K \setminus B(u_i, \varepsilon)) = 0 \quad \text{if } \|u - u_i\| \geqslant \varepsilon.$$

Define a mapping

$$A_\varepsilon : K_\varepsilon \to K_\varepsilon$$
$$A_\varepsilon(u) = P_\varepsilon(A(u))$$

K_ε is a finitely dimensional convex set so it is homeomorphic to a finitely dimensional closed ball and hence by the classical Brouwer fixed point theorem

$$A_\varepsilon(u_\varepsilon) = u_\varepsilon \text{ for some } u_\varepsilon \in K_\varepsilon .$$

Since K is compact and $u_\varepsilon \in K_\varepsilon \subset K$, there is a sequence $\varepsilon_j \to 0$ such that $u_{\varepsilon_j} \to u \in K$. We claim that $A(u) = u$. We have

$$\| u_{\varepsilon_j} - A(u_{\varepsilon_j}) \| = \| A_{\varepsilon_j}(u_{\varepsilon_j}) - A(u_{\varepsilon_j}) \| =$$

$$\| P_{\varepsilon_j}(A(u_{\varepsilon_j})) - A(u_{\varepsilon_j}) \| \le \varepsilon_j$$

$$\| \underset{\substack{\downarrow \\ u}}{u_{\varepsilon_j}} - \underset{\substack{\downarrow \\ A(u)}}{A(u_{\varepsilon_j})} \| \le \varepsilon_j \longrightarrow 0$$

(A is continuous)

$$A(u) = u .$$

<u>Definition</u> A nonlinear operator $A : X \to X$ is compact if for every bounded sequence $\{u_k\}_k \subset X$, $\{A(u_k)\}_k \subset X$ has a convergent subsequence.

<u>Corollary</u> If $A : X \to X$ is continuous and compact and

$$A : \overline{B}(0,R) \longrightarrow \overline{B}(0,R)$$

then A has a fixed point in $\overline{B}(0,R)$.

Proof. $\overline{A(\overline{B}(0,R))}$ is compact so is the closed convex hull

$$K = \overline{co}\left(\overline{A(\overline{B}(0,R))}\right) \subset \overline{B}(0,R)$$

and $A : K \to K$ is continuous so A has a fixed point in $K \subset \overline{B}(0,R)$. $\square$

Theorem (Schaefer's Fixed Point Theorem)

Suppose $A : X \to X$ is continuous and compact. Assume further that the set

$$\{u \in X \mid u = \lambda A(u) \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then A has a fixed point.

Proof. Choose $M > 0$ such that

(*) $\|u\| < M$ if $u = \lambda A(u)$ for some $0 \leq \lambda \leq 1$

and define

$$\tilde{A}(u) = \begin{cases} A(u) & \text{if } \|A(u)\| \leq M \\ \dfrac{M A(u)}{\|A(u)\|} & \text{if } \|A(u)\| \geq M \end{cases}$$

Then $\tilde{A}$ is continuous and compact and

$$\hat{A} : \overline{B}(0,R) \longrightarrow \overline{B}(0,R).$$

Thus it follows from the Corollary that $\tilde{A}$ has a fixed point.

$$\tilde{A}(u) = u.$$

It remains to prove that $A(u) = u$. Suppose to the contrary that $A(u) \neq u$ so $A(u) \neq \tilde{A}(u)$. Then $\|A(u)\| > M$ and

$$u = \tilde{A}(u) = \frac{M\, A(u)}{\|A(u)\|} = \lambda A(u), \qquad \lambda = \frac{M}{\|A(u)\|} < 1.$$

Since $\|u\| = \|\tilde{A}(u)\| = M$ we arrive to a contradiction with (*). $\square$

Now we will show how Schaefer's theorem applies to prove existence of solutions to some non-linear elliptic PDE's.

Consider the boundary value problem

$$(*) \quad \begin{cases} -\Delta u + b(\nabla u) + \mu u = 0 & \text{in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

We assume that Ω is a bounded domain with $\partial\Omega \in C^2$. We also assume that b is a smooth function that satisfies the estimate

$$|b(\xi)| \leq C\,(|\xi| + 1) \quad \text{for all } \xi \in \mathbb{R}^n.$$

Theorem There is $\gamma > 0$ such that for all $\mu \geq \gamma$, there is $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ satisfying $(*)$.

Proof If $u \in W_0^{1,2}(\Omega)$, then the growth condition for b implies that

$$f = -b(\nabla u) \in L^2(\Omega).$$

It follows from the Lax-Milgram theorem that if $\mu \geq 0$ then the boundary value problem

$$\begin{cases} -\Delta w + \mu w = f \\ w \in W_0^{1,2}(\Omega) \end{cases}$$

has a unique solution. By regularity theory this unique solution satisfies

$$\|w\|_{W^{2,2}(\Omega)} \leq c \, \|f\|_{L^2(\Omega)}$$

Note that w is uniquely determined by u and we can define a non-linear operator $Au = w$.

We have

$$\|Au\|_{W^{2,2}(\Omega)} = \|w\|_{W^{2,2}} \le C\|f\|_2 =$$

$$= C\|b(\nabla u)\|_2 \le c\,\||\nabla u|+1\|_2 \le c\left(\|u\|_{W_0^{1,2}(\Omega)}+1\right)$$

$$(*) \qquad \|Au\|_{W^{2,2}(\Omega)} \le c\left(\|u\|_{W_0^{1,2}(\Omega)}+1\right)$$

<u>Claim</u> $A: W_0^{1,2}(\Omega) \longrightarrow W_0^{1,2}(\Omega)$ is continuous and compact.

<u>Compactness</u> If $\{u_k\}_k$ is a bounded sequence in $W_0^{1,2}(\Omega)$ then $(*)$ implies that

$$\sup_k \|Au_k\|_{W^{2,2}(\Omega)} < \infty,$$

so compactness of embedding $W^{2,2}(\Omega) \subset\subset W^{1,2}(\Omega)$, $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \subset\subset W_0^{1,2}(\Omega)$ implies that Au_k has a subsequence convergent in $W_0^{1,2}(\Omega)$.

<u>Continuity</u> Let $u_k \to u$ in $W_0^{1,2}(\Omega)$. We need to show that $A(u_k) \to A(u)$ in $W_0^{1,2}(\Omega)$.

To this end it suffices to show that every subsequence $\{u_{k_j}\}$ has a further subsequence $u_{k_{j_\ell}}$ such that $A(u_{k_{j_\ell}}) \to A(u)$. Take any subsequence of $\{u_k\}$ and still denote it by $\{u_k\}$.

(*) implies that

$$\sup_k \| \underbrace{A(u_k)}_{\omega_k} \|_{W^{2,2}} < \infty$$

Compactness of the embedding $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \subset\subset W_0^{1,2}(\Omega)$ yields that there is a subsequence $\{u_{k_j}\}$ such that $A(u_{k_j}) = \omega_{k_j} \to \omega$ in $W_0^{1,2}(\Omega)$ and it remains to show that $\omega = A(u)$.

Since ω_{k_j} is a weak solution to

$$\begin{cases} -\Delta \omega_{k_j} + \mu \omega_{k_j} = -b(\nabla u_{k_j}) \\ \omega_{k_j} \in W_0^{1,2}(\Omega) \end{cases}$$

we have

$$(**) \quad \int_\Omega \nabla \omega_{k_j} \cdot \nabla \varphi + \mu \omega_{k_j} \varphi \, dx = -\int_\Omega b(\nabla u_{k_j}) \varphi$$

for all $\varphi \in W_0^{1,2}(\Omega)$.

Since

$$\omega_{k_j} \to \omega \text{ in } W_0^{1,2}(\Omega)$$
$$\nabla u_{k_j} \to \nabla u \text{ in } L^2(\Omega)$$
$$b(\nabla u_{k_j}) \to b(\nabla u) \text{ in } L^2(\Omega)$$

passing to the limit in $(**)$ yields

$$\int_\Omega \nabla w \cdot \nabla \varphi + \mu w \varphi = - \int_\Omega b(\nabla u) \varphi$$

i.e.

$$-\Delta w + \mu w = - b(\nabla u)$$

$$w = A(u).$$

This completes the proof of continuity.

<u>Claim</u> If A has a fixed point, $Au = u$, then $u \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}_0(\Omega)$ is a solution to

$$-\Delta u + b(\nabla u) + \mu u = 0$$

Indeed, $u = Au = w \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$

$$-\Delta w + \mu w = - b(\nabla u)$$

$$-\Delta u + \mu u = - b(\nabla u).$$

Thus it remains to show that A has a fixed point. so we need to show that A satisfies the assumptions of the Scheafer theorem.

To this end it suffices to show that if $\mu \geq \gamma$, where γ is sufficiently large, then the set

$\{ u \in W_0^{1,2}(\Omega) \mid u = \lambda A(u) \text{ for some } 0 \le \lambda \le 1 \}$

is bounded in $W_0^{1,2}(\Omega)$.

Thus assume that

$$u = \lambda A(u), \qquad 0 < \lambda \le 1$$

Then

$$\frac{u}{\lambda} = A(u) \in W^{2,2} \cap W_0^{1,2}$$

i.e.

$$-\Delta \frac{u}{\lambda} + \mu \frac{u}{\lambda} = -b(\nabla u)$$

$$-\Delta u + \mu u = -\lambda b(\nabla u)$$

Taking $u \in W_0^{1,2}(\Omega)$ as a test function in the weak formulation of this equation yields

$$\int_\Omega |\nabla u|^2 + \mu |u|^2 = -\int_\Omega \lambda b(\nabla u) u \le$$

$$\le C \int_\Omega (|\nabla u| + 1)|u| \le$$

$$\le C \int_\Omega \underbrace{|\nabla u||u|}_{\le \varepsilon |\nabla u|^2 + C(\varepsilon)|u|^2} + C \int_\Omega \underbrace{|u|}_{\le |u|^2 + 1}$$

$$\le \frac{1}{2} \int_\Omega |\nabla u|^2 + C \int_\Omega |u|^2 + 1$$

Taking $\gamma > C$, $\mu \ge \gamma$ we obtain

$$\|u\|_{W_0^{1,2}(\Omega)} \le C$$

for $C > 0$ independent of λ. The proof is complete.