Calculus of Variations: homework # 3
Due day: September 22, 2014

NAME (print):

Circle the problems that you have solved:

9 10 11 12 13 14

The solutions must be written in a **legible** form. The front page **must** be returned. All the papers **must** be stapled. If any of the conditions will not be satisfied, the homework will be disregarded. The homework **will not** be returned to the students.

Problem 9. Prove that the characteristic function of a ball χ_B does not belong to $W_0^{1,p}(B)$ (but certainly it belongs to $W^{1,p}(B)$). **Hint:** *Use the Poincaré inequality.*

Problem 10. Prove that $u(x) = \log|\log|x|| \in W^{1,n}(B^n(0, e^{-1}))$.

Problem 11. Prove that if $u \in W^{1,p}(\Omega)$ and $u = 1$ on a measurable set $E \subset \Omega$, then $\nabla u = 0$ a.e. on E.

Problem 12. Prove that if $u \in W^{1,p}(0,1)$, then u uniquely extends to a continuous function on $[0,1]$. **Hint:** *Uniform continuity.*

Problem 13. Prove that if $|\Omega| < \infty$ and $1 \le p < q < \infty$, then

$$\Big(\fint_\Omega |u|^p\,dx\Big)^{1/p} \le \Big(\fint_\Omega |u|^q\,dx\Big)^{1/q}.$$

Problem 14. Prove that if $|\Omega| < \infty$ and $u \in L^p(\Omega)$, then

$$\frac{1}{2}\Big(\fint_\Omega |u-u_\Omega|^p\,dx\Big)^{1/p} \le \inf_{c\in\mathbb{R}} \Big(\fint_\Omega |u-c|^p\,dx\Big)^{1/p} \le \Big(\fint_\Omega |u-u_\Omega|^p\,dx\Big)^{1/p}.$$

(The right inequality is obvious, so you need only to prove the left one.)