Calculus of Variations: homework # 4
Due day: September 29, 2014

NAME (print):

Circle the problems that you have solved:

15 16 17 18 19 20

The solutions must be written in a **legible** form. The front page **must** be returned. All the papers **must** be stapled. If any of the conditions will not be satisfied, the homework will be disregarded. The homework **will not** be returned to the students.

Problem 15. Prove that $C^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$, $1 \le p < \infty$, $m \ge 1$. **Hint:** *Use convolution.*

Problem 16. Prove that $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$, $1 \le p < \infty$, $m \ge 1$ and hence $W_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$. **Hint:** *Use Problem 15 and a cut-off function* $\eta_R(x) = \eta(x/R)$.

Problem 17. Prove that $C_0^\infty(\mathbb{R}^n)$ is not dense in $W^{1,\infty}(\mathbb{R}^n)$.

Problem 18. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Prove that there is no extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that

$$\|\nabla Eu\|_{L^p(\mathbb{R}^n)} \le C\|\nabla u\|_{L^p(\Omega)}, \quad u \in W^{1,p}(\Omega).$$

Problem 19. Prove that if $u, v \in W^{1,\frac{2n}{n+1}}(\mathbb{R}^n)$, then $uv \in W^{1,1}(\mathbb{R}^n)$.

$C^{0,\alpha}(K)$, for a compact set $K \subset \mathbb{R}^n$, is a Banach space with the norm

$$\|u\|_{C^{0,\alpha}} = \sup_K |u| + \sup_{x,y\in K, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}\,.$$

Problem 20. Prove that for $0 < \alpha < \beta \le 1$ the embedding $C^{0,\beta}([0,1]) \subset C^{0,\alpha}([0,1])$ is compact.