NAME (print):

Circle the problems that you have solved:

1 2 3 4 5 6 7

The solutions must be written in a **legible** form. The front page **must** be returned. All the papers **must** be stapled. If any of the conditions will not be satisfied, the homework will be disregarded.

Analysis 4: homework # 1

10 points for each problem.

Problem 1. Prove the Young's inequality: if $1 \le p, q, r \le \infty$, $q^{-1} = p^{-1} + r^{-1} - 1$, then

$$\|f * g\|_q \le \|f\|_p \|g\|_r \,.$$

Problem 2. Find $\delta_1 * \delta_b$, where δ_a and δ_b are Dirac measures concentrated at points $a, b \in \mathbb{R}^n$.

Problem 3. Prove that if f is locally integrable and $\lim_{a\to\infty} \int_0^a f(x)\, dx = \ell$, then $A_\varepsilon = \int_0^\infty f(x) e^{-\varepsilon x}\, dx$ converges to ℓ as $\varepsilon \to 0^+$.

Problem 4. Evaluate $\int_0^\infty e^{-y^a}\, dy$, $a > 0$. Provide the answer in terms of the Γ function.

Problem 5. The *beta function* is defined by the formula

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx, \quad a, b > 0.$$

Prove that

(a)

$$B(a,b) = 2\int_0^{\pi/2} \sin^{2a-1}\theta \cos^{2b-1}\theta\, d\theta\,,$$

(b)

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,.$$

Problem 6. Prove that

$$e^{-\beta} = \frac{2}{\pi}\int_0^\infty \frac{\cos\beta x}{1+x^2}\, dx, \quad \text{for } \beta > 0.$$

Problem 7. Assume $f \in L^1 \cap L^\infty$ and $\hat{f} \ge 0$. Prove that $\hat{f} \in L^1$.