TSCS Analysis

Binary Outcome TSCS Models

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Spatiotemporal Multivariate Qualitative Models

Important questions: (observed) recursive? (latent) simultaneous? mixed process?

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y_1^* &= \beta_0 + \beta_1 x_1 + \beta_2 y_2 + \beta_3 y_2^* + u \\
y_2^* &= \gamma_0 + \gamma_1 x_1 + \gamma_2 y_1 + \gamma_3 y_1^* + v, \text{ where } (u, v) \sim \mathcal{N}(0, \Sigma)
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4. If $\beta_3 \neq 0$ and $\gamma_3 \neq 0$ simultaneous equation model

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$$y^* = \beta_1 x + \beta_2 y + u,$$

is logically inconsistent unless $\beta_2 = 0$. To see, when $y = 1$ we have $\beta_1 x + \beta_2 < u$ and when $y = 0$ then $\beta_1 x \geq u$, where the probabilities only sum to 1 when $\beta = 0$.

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More general than it might seem, consider

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Only (logically) consistent if $\gamma_1 \beta_1 = 0$

If, however, restrictions are made to ensure that the model is recursive:

$$y_1^* = \beta_0 + \beta_1 x_1 + \beta_2 y_2 + u$$
$$y_2^* = \gamma_0 + \gamma_1 x_1 + v, \text{ where } (u, v) \sim \mathcal{N}(0, \Sigma),$$

then estimation proceeds without issue any additional fixes.

Spatiotemporal Multivariate Qualitative Models

Thus, for simplicity, let's assume $\beta_2 = 0$ and just focus on the correlated errors. To estimate such a model we need to denote a joint distribution of (u, v) – say $F(\cdot, \cdot)$ – and we can define joint probabilities for the outcome profiles:

$$P_{11} = \Pr(y_1 = 1, y_2 = 1) = F[(\beta_1 x_1, \beta_2 x_2); \rho]$$
$$P_{10} = \Pr(y_1 = 1, y_2 = 0) = F[(\beta_1 x_1, -\beta_2 x_2); \rho]$$
$$P_{01} = \Pr(y_1 = 0, y_2 = 1) = F[(-\beta_1 x_1, \beta_2 x_2); \rho]$$
$$P_{00} = \Pr(y_1 = 0, y_2 = 0) = F[(-\beta_1 x_1, -\beta_2 x_2); \rho]$$

Which we could specify a likelihood function for as:

$$L(\beta_1, \beta_2) = \prod P_{11}^{y_1 y_2} P_{11}^{y_1(1-y_2)} P_{11}^{(1-y_1)y_2} P_{11}^{(1-y_1)(1-y_2)}$$

Spatiotemporal Multivariate Qualitative Models

To back up for a second, maximizing the likelihood from the last slide involves the evaluation of double integrals (the bivariate normal CDF). For example

$$P_{00} = \int_{-\infty}^{-\beta_1 x_1} \int_{-\infty}^{-\beta_2 x_2} \phi((u,v)'; \Sigma) du dv = \Phi(-\beta_1 x_1, -\beta_2 x_2; \rho),$$

or

$$P_{11} = \int_{-\beta_1 x_1}^{\infty} \int_{-\beta_2 x_2}^{\infty} \phi((u,v)'; \Sigma) du dv = \Phi(\beta_1 x_1, \beta_2 x_2; \rho),$$

Can generalize using $q_j = 2y_{ij} - 1$

$$\ell = \sum \log \Phi(q_j \beta_1 x_1, q_j \beta_2 x_2; q_1 q_2 \rho),$$

Space-Time Dependence via Latents

This represents a “pure” strategy to model autoregressive dependence in space and time (Franseze, Hays, Cook 2016). However. . .

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Therefore we discuss some simple strategies (Beck, Katz, Tucker 1998; Carter and Signorino 2010)

On time: BTSCS is duration data

Underlying data structure:

t	1	2	3	4	5	6	7
y	0	0	1	1	1	1	0

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Therefore we can use strategies from duration models to account for time dependence in repeated-measure binary outcome data

Discrete-Time Hazard Models

- Discrete-time hazard models are used when the time to an event is either inherently discrete or group into discrete intervals of time ("interval censoring").
- With interval censoring, we model the probability that an event occurs within a particular interval with logit or probit models.
- The baseline hazard (i.e., the hazard function when all covariates are set to zero) can be modeled using either time-period dummies, splines, or polynomials.
- Polynomials in time are the easiest method to implement and interpret.
  - Time-period dummies are the most flexible, but they eat up degrees of freedom and commonly suffer from separation problems.
  - Splines are more difficult to implement correctly and rarely interpreted.
- Typically a cubic polynomial in time is sufficient to capture the baseline hazard.

Underlying maths: failure distribution

Define the CDF as

$$F(t) = \int_0^t f(u)d(u) = \text{pr}(T \leq t),$$

which is the probability that survival time T is less than or equal to t (sometimes called the failure distribution).

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$$f(t) = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} = \frac{\text{pr}(t \leq T \leq (t + \Delta t))}{\Delta t},$$

which is the conditional (or instantaneous) failure rate, that is, the probability that failure (an event) occurs within an infinitesimally small time.

Underlying maths: survivor function

The complement to the failure distribution is the, wait for it, survivor function:

$$S(t) = 1 - F(t) = \mathsf{pr}(T \geq t),$$

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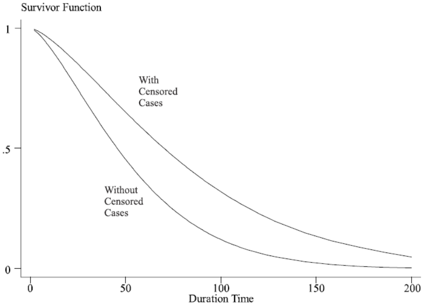

Underlying maths: hazard rate

Now we have the parts necessary to define the risk of failure, that is, the hazard rate

$$h(t) = \frac{f(t)}{S(t)} = \frac{\text{pr}(t \leq T \leq (t + \Delta t)|T \leq t)}{\Delta t}$$

which is the conditional failure rate – e.g., Given that the US has been a democracy since 1776, what are the chances it will transition to autocracy in 2017?

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- Polynomials in time are the easiest method to implement and interpret.
  - time-since-event counter, time-since-event counter^2, time-since-event counter^3
- Typically a cubic polynomial in time is sufficient to capture the baseline hazard.

Regime (State) Switching Models

- When regimes (or states) are observed and persist over time, it is relatively straightforward to estimate a switching model.
- In the two-regime case, we use logit or probit to model the probability of transitioning from one regime to the the other, or more generally $\Pr(S_t = 1|S_{t-1}, X_{t-1})$.
- On the right-hand side, the regression includes the regime at time $t-1$, covariates, and covariates interacted with the regime at time $t-1$, for example, in the simplest case

$$y*_t = \beta_0 + \beta_1 S_{t-1} + \beta_2 x_{t-1} + \beta_3 S_{t-1} \times x_{t-1},$$

where $y*_t$ is a latent variable that determines the probability of observing a one at time t.

Regime (State) Switching Models

$$y*_t = \beta_0 + \beta_1 S_{t-1} + \beta_2 x_{t-1} + \beta_3 S_{t-1} \times x_{t-1},$$

In the case of logit,

- The probability of persisting in the regime coded zero P_{00} is

$$1 - \frac{1}{1 + \exp^{-(\beta_0 + \beta_2 x_{t-1})}}.$$

- The probability of switching to the regime coded one is $1 - P_{00}$.
- The probability of persisting in the regime coded one P_{11} is

$$\frac{1}{1 + \exp^{-([\beta_0 + \beta_1] + [\beta_2 + \beta_3] x_{t-1})}}.$$

- The probability of switching to the regime coded zero is $1 - P_{11}$.

On space: Spatial Filtering

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But! Spatial filtering is extremely flexible and can be used here as well. Much better alternative than using the spatially lagged observed value.

On heterogeneity: RE vs. FE

Many of the same considerations as with interval-valued outcomes (i.e., heterogeneity in the intercepts, orthogonality of the unit effects and the predictors, distribution of the unit effects, etc. . .).

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Some additional complications with fixed effects however:

- Cannot demean the data for fixed effects (therefore, incidental parameters problem)
- Unconditional vs. Conditional
- Sample Censoring

On heterogeneity: PML-FE

Consequence of sample censoring:

$$\underbrace{\Pr(Y=1)}_{\text{Population Relationship}} = \underbrace{\Pr(Y=1|Z=1)}_{\text{Uncensored Relationship}} \underbrace{\Pr(Z=1)}_{\text{Probability Uncensored}} + \underbrace{\Pr(Y=1|Z=0)}_{\text{Censored Relationship}}(1-\Pr(Z=1))$$

$$\text{AME} = \frac{1}{n}\sum_{i=1}^{n} \Pr(Y \widehat{= 1|x_i}, \hat{\beta}, \hat{\alpha}_i) \times (1 - \Pr(Y \widehat{= 1|x_i}, \hat{\beta}, \hat{\alpha}_i)) \times \hat{\beta}$$

On heterogeneity: PML-FE

- Penalized Maximum Likelihood due to Firth (1993)

$$L^*(\theta) = L(\theta)|I(\theta)|^{\frac{1}{2}}.$$

- Shown to be a solution for separation (Heinze and Schempher 2001)
- Sample censoring induced from unconditional fixed effects is separation

Therefore you can using PML to estimate a fixed effects model with binary outcomes (Cook, Hays, Franzese 2018)

In Sum

So often our recommendation for a ‘simple’ model with binary-TSCS data would be a PML-FE model with cubic polynomials and spatial filtering.