Time Series Analysis

Cointegration and Error-Correction Models (Enders, Chapter 6)

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Outline

1. What is Cointegration?
2. Cointegration and Error-Correction Models
3. Testing for Cointegration

Cointegration Defined

- A set of integrated variables is said to be *cointegrated* when a stationary linear combination of these variables exists.
- Theories that imply equilibrium relationships among nonstationary variables require cointegration. We can represent this long-run equilibrium relationship as

$$\beta_1 x_{1t} + \beta_2 x_{2t} + \ldots + \beta_n x_{nt} = 0 \text{ or } \beta \mathbf{x_t} = 0$$

  where $e_t = \beta \mathbf{x_t}$ is the *equilibrium error*.
- Engel and Granger (1987) provide a more formal definition of *cointegration*: the elements of the vector $\mathbf{x_t}$ are *cointegrated of order d,b*, $\mathbf{x_t} \sim CI(d, b)$, if

1. All elements of $\mathbf{x_t}$ are integrated of order d.
2. There exists a vector $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ such that the linear combination is integrated of order (d-b), where $b > 0$.

Cointegration Defined

- Note that the conintegrating vector, $(\beta_1, \beta_2, ..., \beta_n)$, is not unique. For any non-zero scalar λ, $(\lambda\beta_1, \lambda\beta_2, ..., \lambda\beta_n)$ is also a cointegrating vector.
- With this in mind, we typically normalize the cointegrating vector so that one of the coefficients is one. For example, with $\lambda = 1/\beta_1$, we have $(1, \beta_2/\beta_1, ..., \beta_n/\beta_1)$.
- If $\mathbf{x_t}$ has n non-stationary elements, there are at most $n-1$ cointegrating vectors.
- Cointegration is a relationship among variables that are integrated of the same order. Most of the time we are referring to the $CI(1,1)$ case.

Cointegration and Common Trends

- Cointegrated variables share a common stochastic trend (Stock and Watson, 1988). To see this, consider the case where $\mathbf{x_t} = (y_t, z_t)$ such that

$$\begin{aligned} y_t &= \mu_{yt} + e_{yt} \\ z_t &= \mu_{zt} + e_{zt} \end{aligned}$$

  where the $\mu_{\mathbf{t}}$ are stochastic trends and the $\mathbf{e_t}$ are stationary.
- Cointegration implies that linear combination of y_t and z_t is stationary. Given

$$\begin{aligned} \beta_1 y_t + \beta_2 z_t &= \beta_1(\mu_{yt} + e_{yt}) + \beta_2(\mu_{zt} + e_{zt}) \\ &= (\beta_1 \mu_{yt} + \beta_2 \mu_{zt}) + (\beta_1 e_{yt} + \beta_2 e_{zt}) \end{aligned}$$

  which implies that $\beta_1 \mu_{yt} + \beta_2 \mu_{zt} = 0$ or $\mu_{yt} = -\beta_2 \mu_{zt}/\beta_1$.
- This easily generalizes to the n-variable case any time one stochastic trend can be expressed as a linear combination of the other trends.

Error-Correction Models

- Cointegration implies an equilibrium relationship and defines an equilibrium "error."
- For the equilibrium to be substantively meaningful, this error process must be stationary. That is, deviations from the equilibrium must be temporary.
- We represent this kind of relationship with an error-correction model (ECM), which in the simplest case is

$$\Delta y_t = \alpha_y(y_{t-1} - \beta z_{t-1}) + \varepsilon_{yt}$$
$$\Delta z_t = \alpha_z(y_{t-1} - \beta z_{t-1}) + \varepsilon_{zt}$$

  where $(y_{t-1} - \beta z_{t-1})$ is the equilibrium error, the α are adjustment parameters, and the $\varepsilon_{\mathbf{t}}$ are white noise.
- Note that if $\alpha_y < 0$ and $\alpha_z > 0$ and y is above its long-run equilibrium value relative to z, y will decrease and z to "correct the error" and restore equilibrium.

Error-Correction Models

- Note that an error-correction model is just a restricted VAR. To see this, start with the simple VAR

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}$$

- Subtracting $\mathbf{x_t} = (y_t, z_t)'$ from both sides gives

$$\begin{bmatrix} \Delta y_t \\ \Delta z_t \end{bmatrix} = \begin{bmatrix} a_{11} - 1 & a_{12} \\ a_{21} & a_{22} - 1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}$$

- Imposing the restriction $a_{11} - 1 = -a_{12}a_{21}/(1 - a_{22})$, and defining

$$\alpha_y = -a_{12}a_{21}/(1 - a_{22})$$
$$\beta = (1 - a_{22})/a_{21}$$
$$\alpha_z = a_{21}$$

gives us the simple ECM.

Error-Correction Models

- Consider again the system written in terms of $\Delta \mathbf{x_t}$

$$\begin{bmatrix} \Delta y_t \\ \Delta z_t \end{bmatrix} = \begin{bmatrix} a_{11} - 1 & a_{12} \\ a_{21} & a_{22} - 1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}$$

- Define $\pi = (\mathbf{A_1} - \mathbf{I})$ such that $\Delta \mathbf{x_t} = \pi \mathbf{x_{t-1}} + \varepsilon_{\mathbf{t}}$.
- Note that the restrictions which imply cointegration and error-correction create linear dependence among the rows of π.
- This fact will be important later. It suggests that we can use the rank of π, the number of linearly independent rows, to identify cointegration.

The Engle-Granger Methodology

There are several steps in the Engle-Granger approach to cointegration. Let's consider the simple case of two variables, y_t and z_t.

- Identify the order of integration for each of the variables using the appropriate Dickey-Fuller tests. If the variables are stationary, you do not *need* an error correction model (although you might choose one anyway). If the one of the variables are $I(2)$ or higher, you might consider multicointegration.
- Estimate the long-run equilibrium relationship between y_t and z_t using the regression

$$y_t = \beta_0 + \beta_1 z_t + e_t$$

OLS provides consistent estimates of the cointegrating vector, $\beta = (\beta_0, \beta_1)$.

The Engle-Granger Methodology

- Identify the order of integration for the estimated disturbances $\{\hat{e}_t\}$. Estimate the regression

$$\Delta \hat{e}_t = a_1 \hat{e}_{t-1} + \varepsilon_t,$$

 and test the null hypothesis $a_1 = 0$. In this instance, we cannot use the standard Dickey-Fuller tables to obtain critical values because the disturbances are estimated rather than observed. We need special tables that take this additional source of uncertainty into account.

- If $\{y_t\}$ and $\{z_t\}$ are determined to be $I(1)$ and the disturbances are $I(0)$, we can conclude that $\mathbf{x_t} \sim CI(1, 1)$.

The Engle-Granger Methodology

- Estimate the ECM using the residuals from the equilibrium regression. That is, estimate the system

$$\begin{aligned}\Delta y_t &= \alpha_1 + \alpha_y \hat{e}_{t-1} + \sum \alpha_{11}(i)\Delta y_{t-i} + \sum \alpha_{12}(i)\Delta z_{t-i} + \varepsilon_{yt} \\ \Delta z_t &= \alpha_2 + \alpha_z \hat{e}_{t-1} + \sum \alpha_{21}(i)\Delta y_{t-i} + \sum \alpha_{22}(i)\Delta z_{t-i} + \varepsilon_{zt}\end{aligned}$$

- Note that this is a VAR in first differences. The standard VAR methods can be used to estimate and analyze this system of equations (e.g., lag-length and block-exogeneity tests).
- Assess the model's adequacy (make sure the disturbances are white noise), conduct causality tests (these test have to include the relevant parameter from $\alpha_{\mathbf{x}}$), and do innovation accounting.
- In general, the Engle-Granger approach does not provide a methodology for evaluating hypotheses on the cointegrating vector, although the t-statistics can be adjusted using fully-modified least squares to make inference possible