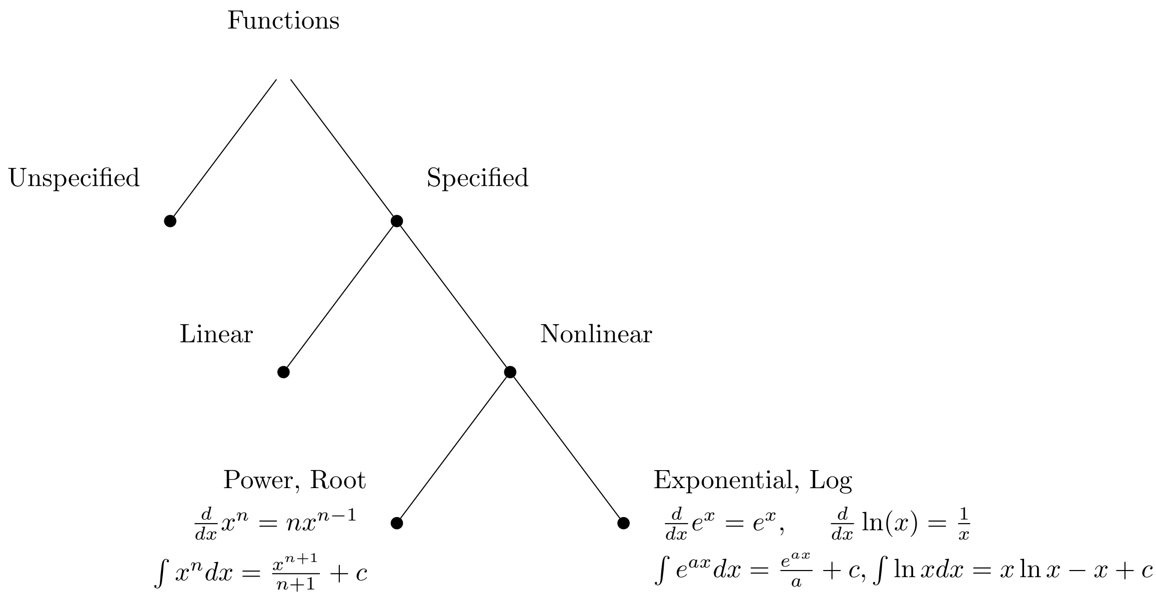
The Calculus of Common Functions

A function is "a relation that assigns one element of the range to each element of the domain."



Some more rules for derivatives:

Product: $\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

Quotient: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2}$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, where $y = f(u), u = g(x)$

Some techniques for integration:

Integration by Substitution: $\int w(u)dx = \int \left[w(u)\frac{dx}{du}\right]du$

Integration by Parts: $\int udv = uv - \int vdu$

Order for choosing u: 1) log function, 2) power function and 3) exponential function.

The Properties of Matrix Operations

Matrix Transposition Properties

- Invertibility Property

$$(\mathbf{X}')' = \mathbf{X}$$

- Additive Property

$$(\mathbf{X}+\mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$$

- Multiplicative Property

$$(\mathbf{XY})' = \mathbf{Y}'\mathbf{X}'$$

- Scalar Multiplication Property

$$(c\mathbf{X})' = c\mathbf{X}'$$

- Inverse Transpose Property

$$(\mathbf{X}^{-1})' = (\mathbf{X}')^{-1}$$

- Symmetric Matrix Property

$$\mathbf{X}' = \mathbf{X}$$

Matrix Multiplication Properties

If the matrices **A**, **B**, **C** are conformable for multiplication, then

- Associative Property

$$(\mathbf{AB})\,\mathbf{C} = \mathbf{A}\,(\mathbf{BC})$$

- Distributive Property

$$\mathbf{A}\,(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$

- Transpose of a Product

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$$

Matrix Determinant Properties

- Transpose Property

$$|\mathbf{X}| = \left|\mathbf{X}'\right|$$

- Diagonal or Triangular Matrix Property ($n \times n$)

$$|\mathbf{X}| = \prod_{i=1}^{n} x_{ii}$$

- Identity Matrix Property ($n \times n$)

$$|\mathbf{I}| = 1$$

- Scalar Multiplication Property ($n \times n$)

$$|c\mathbf{X}| = c^n\,|\mathbf{X}|$$

- Multiplicative Property

$$|\mathbf{XY}| = |\mathbf{X}|\,|\mathbf{Y}|$$

- Inverse Property

$$\left|\mathbf{X}^{-1}\right| = \frac{1}{|\mathbf{X}|}$$

Matrix Inverse Properties

- Diagonal Matrix Property $\mathbf{X}^{-1}$ has diagonal values $1/x_{ii}$ and zeros elsewhere. Therefore, $\mathbf{I}^{-1} = \mathbf{I}$.

- Iterated Inverse Property

$$(\mathbf{X}^{-1})^{-1} = \mathbf{X}$$

- Scalar Multiplication Property

$$(c\mathbf{X})^{-1} = \frac{1}{c}\mathbf{X}^{-1}$$

- Multiplicative Property

$$(\mathbf{XY})^{-1} = \mathbf{Y}^{-1}\mathbf{X}^{-1}$$

The Trace of a Matrix and Its Properties

The trace of a square matrix is the sum of its diagonal values.

$$tr(\mathbf{X}) = \sum_{i=1}^{n} x_{ii}$$

- Identity Matrix Property
$$tr(\mathbf{I}_n) = n$$
- Scalar Multiplication Property
$$tr(c\mathbf{X}) = c \times tr(\mathbf{X})$$
- Matrix Addition Property
$$tr(\mathbf{X}+\mathbf{Y}) = tr(\mathbf{X}) + tr(\mathbf{Y})$$
- Matrix Multiplication Property
$$\mathrm{tr}(\mathbf{XY}) = \mathrm{tr}(\mathbf{YX})$$
- Transposition Property: $tr(\mathbf{X}') = tr(\mathbf{X})$.

Matrix Algebra and Calculus

Calculating the determinant (using row 1):

$$\det(\mathbf{A}) = (-1)^{1+1} \times a_{11} \times \det(\mathbf{A}_{11}) + (-1)^{1+2} \times a_{12} \times \det(\mathbf{A}_{12}) + ... + (-1)^{1+n} \times a_{1n} \times \det(\mathbf{A}_{1n})$$

where $\mathbf{A_{ij}}$ is the submatrix formed by deleting row i and j from $\mathbf{A}$.

Inverting a $n \times n$ matrix

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{C}'$$

where

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}_{11}(-1)^{1+1} & \mathbf{A}_{12}(-1)^{1+2} & \dots & \mathbf{A}_{1n}(-1)^{1+n} \\ \mathbf{A}_{21}(-1)^{2+1} & \mathbf{A}_{22}(-1)^{2+2} & \dots & \mathbf{A}_{21}(-1)^{2+n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{A}_{n1}(-1)^{n+1} & \mathbf{A}_{n2}(-1)^{n+2} & \dots & \mathbf{A}_{nn}(-1)^{n+n} \end{bmatrix}$$

Cholesky's decomposition for $\mathbf{\Sigma} = \mathbf{LL}'$

$$\ell_{jj} = \sqrt{\sigma_{jj} - \sum_{k=1}^{j-1} \ell_{jk}^2}$$
$$\ell_{ij} = \frac{1}{\ell_{jj}}\left(\sigma_{ij} - \sum_{k=1}^{j-1} \ell_{ik}\ell_{jk}\right) \text{for } i > j$$

If we have a multivariate scalar function $f(\mathbf{x}) = f(x_1, x_2...x_m)$, the $(m \times 1)$ **gradient** vector is

$$\nabla f = \begin{bmatrix} f_{x_1} \\ f_{x_2} \\ \vdots \\ f_{x_m} \end{bmatrix}$$

If we have a multivariate scalar function $f(\mathbf{x}) = f(x_1, x_2...x_m)$, the $(m \times m)$ **Hessian** matrix is

$$\nabla^2 f = \mathbf{H} = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_m} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_m} \\ \cdots & \cdots & \cdots & \cdots \\ f_{x_mx_1} & f_{x_mx_2} & \cdots & f_{x_mx_m} \end{bmatrix}$$

If we have a multivariate scalar function $\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1, x_2...x_m)$, the $(n \times m)$ **Jacobian** matrix is

$$\mathbf{J} = \begin{bmatrix} f_{1x_1} & f_{1x_2} & \cdots & f_{1x_m} \\ f_{2x_1} & f_{2x_2} & \cdots & f_{2x_m} \\ \cdots & \cdots & \cdots & \cdots \\ f_{nx_1} & f_{nx_2} & \cdots & f_{nx_m} \end{bmatrix}$$

Optimizing Multivariate Scalar Functions

- The multivariate version of the FOC sets the $(m \times 1)$ gradient vector of first-partial derivatives equal to an $(m \times 1)$ vector of zeros and solves for $\mathbf{x}^*$.
- The multivariate version of the SOC checks to see whether the $(m \times m)$ Hessian matrix of second-partial derivatives and cross-partial derivatives is convex or concave at $\mathbf{x}^*$.
  - If $\mathbf{H}$ is concave, $f(\mathbf{x})$ has a *local* maximum at $\mathbf{x}^*$.
  - If $\mathbf{H}$ is convex, $f(\mathbf{x})$ has a *local* minimum at $\mathbf{x}^*$.
  - The largest local maximum is the *global* maximum, and the smallest local minimum is the *global* minimum.
- How do we know if $\mathbf{H}$ is concave or convex?
  - If the Hessian, $\mathbf{H}$, is negative definite, then the function $f(\mathbf{x})$ is concave at $\mathbf{x}^*$.
  - If the Hessian, $\mathbf{H}$, is positive definite, then the function $f(\mathbf{x})$ is convex at $\mathbf{x}^*$.
- An $n \times n$ matrix $\mathbf{\Sigma}$ is positive definite if the scalar $\mathbf{x}'\mathbf{\Sigma}\mathbf{x}$ is strictly positive for every non-zero column vector $\mathbf{x}$.
- An $n \times n$ matrix $\mathbf{\Sigma}$ is positive definite if and only if all of its eigenvalues, $\boldsymbol{\omega}$, are positive.

Eigenvalues and Eigenvectors

- Let $\mathbf{A}$ be an $N \times N$ matrix. The vector $\mathbf{v}$ is an **eigenvector** of $\mathbf{A}$, if it satisfies

  $$\mathbf{A}\mathbf{v} = \omega\mathbf{v},$$

  for a scalar ω, which is called the **eigenvalue** of $\mathbf{A}$ that corresponds to $\mathbf{v}$.
- The eigenvalues of a square matrix $\mathbf{A}$ are the values of ω that solve

  $$\det(\mathbf{A} - \omega\mathbf{I}_N) = 0$$

- This equation is called the **characteristic equation** of $\mathbf{A}$ and the determinant on the left hand side is called the **characteristic polynomial** of $\mathbf{A}$.