PolSci 2740

Homework # 1

(Show your work)

DUE: September 11, 2024

September 3, 2024

1 Some Spatial Autoregression Math

1.1 Intro to the Spatial Autoregressive (SAR) Model

The spatial-lag model is:

$$\mathbf{y} = \rho\mathbf{W}\mathbf{y} + \mathbf{X}\beta + \varepsilon \Rightarrow \varepsilon = (\mathbf{I} - \rho\mathbf{W})\,\mathbf{y} - \mathbf{X}\beta = \mathbf{A}\mathbf{y} - \mathbf{X}\beta$$

The likelihood for ε is:

$$L(\varepsilon) = \left(\frac{1}{\sigma^2 2\pi}\right)^{N/2} \exp\left(-\frac{\varepsilon'\varepsilon}{2\sigma^2}\right)$$

The inverse function is: $\varepsilon = r^{-1}(\mathbf{y}) = (\mathbf{I} - \rho\mathbf{W})\,\mathbf{y} - \mathbf{X}\beta$

The Jacobian is: $\frac{\partial\varepsilon}{\partial\mathbf{y}} = (\mathbf{I} - \rho\mathbf{W}) = \mathbf{A}$

By the multivariate change of variables theorem, the likelihood for $\mathbf{y}$ is

$$L(\mathbf{y}) = |\mathbf{A}| \left(\frac{1}{\sigma^2 2\pi}\right)^{N/2} \exp\left(-\frac{1}{2\sigma^2}(\mathbf{A}\mathbf{y} - \mathbf{X}\beta)'\,(\mathbf{A}\mathbf{y} - \mathbf{X}\beta)\right)$$

The log-likelihood function is

$$LL(\mathbf{y}) = \ln|\mathbf{A}| + \left(\frac{N}{2}\right)\ln\left(\frac{1}{2\pi\sigma^2}\right) + \left(-\frac{1}{2\sigma^2}(\mathbf{A}\mathbf{y} - \mathbf{X}\beta)'\,(\mathbf{A}\mathbf{y} - \mathbf{X}\beta)\right)$$

1.2 Estimating the SAR Model

The gradient of the log-likelihood function is equal to the zero vector at a stationary point.

$$G = \nabla LL = \begin{bmatrix} LL_\rho \\ LL_\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using a numerical algorithm (e.g., Newton-Raphson, Berndt-Hall-Hall-Hausman, Davidon-Fletcher-Powell), you find the stationary point at

$$\begin{bmatrix} \hat{\rho} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} .09 \\ 1.5 \end{bmatrix}$$

If the stationary point is a maximum, the matrix of second derivatives, the Hessian matrix, will be negative definite.

$$H = \nabla^2 LL = \begin{bmatrix} LL_{\rho\rho} & LL_{\rho\beta} \\ LL_{\beta\rho} & LL_{\beta\beta} \end{bmatrix} = \begin{bmatrix} -692 & -10 \\ -10 & -3 \end{bmatrix}$$

a. Check that the Hessian is negative definite by solving for its eigenvalues.

b. Calculate the (asymptotic) variance-covariance matrix using the Fisher information matrix, $(-H)^{-1}$.

2 Difference Equations

Enders, Exercise #13, p.45.

Consider the following two stochastic difference equations:

i $y_t = 3 + 0.75y_{t-1} - 0.125y_{t-2} + \varepsilon_t$

ii $y_t = 3 + 0.25y_{t-1} - 0.375y_{t-2} + \varepsilon_t$

a. Use the method of undetermined coefficients to find the particular solution for each equation.

b. Find the homogeneous solutions for each equation.

c. For each process, suppose that $y_0 = y_1 = 8$ and that all values of ε_t for $t = 1, 0, -1, -2, ... = 0$. Use the method illustrated by equations (1.75) and (1.76) to find the values of A_0 and A_1.

   Also, calculate the first two impulse responses for ε_t, $\frac{\partial y_{t+1}}{\partial \varepsilon_t}$ and $\frac{\partial y_{t+2}}{\partial \varepsilon_t}$.