DIFFERENTIAL GEOMETRY 2, HOMEWORK 6 ADDENDUM

(1) (*Riemannian Manifolds*, Problem 8-8) Suppose $g = g_1 \times g_2$ is a product metric on $M = M_1 \times M_2$ as in Kühnel's Chapter 5, Exercise 6.
   (a) Show that for any $p_i \in M_i$, the submanifolds $S_1 = M_1 \times \{p_2\}$ and $S_2 = \{p_1\} \times M_2$ are *totally geodesic*: for any point x in S_i and $\mathbf{v} \in T_x S_i$, the geodesic $c_{\mathbf{v}}$ through x in the direction of $\mathbf{v}$ lies in S_i.
   (b) For $p = (p_1, p_2) \in M$, if $\sigma \subset T_p M$ is spanned by $\mathbf{X}_1 \in T_{p_1} M_1$ and $\mathbf{X}_2 \in T_{p_2} M_2$ then show that $K_\sigma = 0$.
   (c) Show that the product metric on $S^2 \times S^2$ has nonnegative sectional curvature.
   (d) Show that there is an embedding of $S^1 \times S^1$ in $S^2 \times S^2$ such that the induced metric is *flat*; ie. has vanishing curvature.

(2) Compute the sectional curvatures of the metric $g = \frac{1}{(x^{n+1})^2} g_E$ on $\mathbb{R}^n \times (0, \infty)$, where g_E is the standard inner product on $\mathbb{R}^{n+1}$.
Hint: The pullback of g under the diffeomorphism $f \colon \mathbb{R}^{n+1} \to \mathbb{R}^n \times (0, \infty)$ given by $f(x^1, \ldots, x^n, t) = (x^1, \ldots, x^n, e^t)$ is a warped product:
$$f_* g = e^{-2t} \, g_E + dt^2,$$
where now g_E refers to the standard inner product on $\mathbb{R}^n$.

(3) (*Riemannian Geometry*, Exercise 4.6) A Riemannian manifold (M, g) is a *locally symmetric space* if its Riemann curvature tensor R satisfies $\nabla R = 0$.
   (a) For a locally symmetric space M, a geodesic $c \colon [0, L] \to M$, and parallel vector fields $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ along c, show that $R(\mathbf{X}, \mathbf{Y})\mathbf{Z}$ is also parallel along c.
   (b) Prove that if M is locally symmetric, connected, and of dimension two then M has constant sectional curvature.
   (c) Prove that if M has constant sectional curvature then it is locally symmetric.