April 2012, Problem 3. Suppose that $f\colon (0,1) \to \mathbb{R}$ is continuous and bounded such that the limit $\lim_{x\to 0^+} f(x)$ does not exist. Prove that there is an interval $[a,b]$, $a<b$ such that for every $x \in [a,b]$ there is a decreasing sequence $x_k \in (0,1)$, $x_k \to 0$ with $f(x_k) \to x$.

(*Definition:* $L = \lim_{x\to 0^+} f(x)$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $0 < x < \delta \Rightarrow |f(x) - L| < \epsilon$.)

Hint: You may want to use the fact that $\lim_{x\to 0^+} f(x)$ exists if and only if

$$\limsup_{x\to 0^+} f(x) \doteq \lim_{x\to 0^+} \sup\{f(y) \,|\, 0 < y \leq x\}$$

is equal to $\liminf_{x\to 0^+} f(x)$ (which is defined analogously). If you use it, prove it.