TOPOLOGY 2 - HOMEWORK 1

(1) Prove the following result:

Lemma. *Let X and Y be topological spaces, and suppose $X^* = X/\sim$ is a quotient space of X. A continuous map $f\colon X \to Y$ with the property that $f(x) = f(x')$ whenever $x \sim x'$ determines a unique continuous map $f^*\colon X^* \to Y$ so that the diagram below commutes.*

$$\begin{array}{ccc} X & & \\ {\scriptstyle p}\downarrow & \searrow^{f} & \\ X^* & \xrightarrow{f^*} & Y \end{array}$$

That is, $f = f^* \circ p$, where $p\colon X \to X^*$ is the quotient map. If X is compact, Y is Hausdorff, and f^* is bijective (ie, one-to-one and onto) then f^* is a homeomorphism.

(Hint: For the second part, prove and then use the following fact: any continuous map from a compact space X to a Hausdorff space Y takes closed subsets of X to closed subsets of Y.)

(2) Prove that $S^1 \times S^1$ is homeomorphic to $[0,1] \times [0,1]/\{(0,t) \sim (1,t), (t,0) \sim (t,1) \text{ for all } t \in [0,1]\}$.

(3) Prove that S^n is homeomorphic to the unique CW-complex with a single 0-cell, a single n-cell, and no others.

(Hint: A very useful map is the *stereographic projection* $\mathbb{R}^n \to S^n$, given by:

$$\mathbf{x} \mapsto \left(\frac{2}{\|\mathbf{x}\|^2 + 1}\mathbf{x}, \frac{\|\mathbf{x}\|^2 - 1}{\|\mathbf{x}\|^2 + 1} \right)$$

Compose this with a map taking the interior of $\mathbb{D}^n$ onto $\mathbb{R}^n$ to obtain a map $\mathbb{D}^n \to S^n$.)

(4) Prove that every convex subset of $\mathbb{R}^n$ is contractible.

(5) Prove that S^∞ is contractible. (Hint: For each $n > 0$, the inclusion map $S^{n-1} \times \{0\} \to S^n$ is homotopic in S^n to a constant map.)