TOPOLOGY 2 - HOMEWORK 6

(1) For a group $G = \langle a_1, \ldots, a_n \,|\, \mathcal{R} \rangle$ (where $\mathcal{R}$ is some set of relations), show that:

$$G^{ab} = \langle a_1, \ldots, a_n \,|\, \mathcal{R} \cup \{[a_i, a_j] \,|\, 1 \leq i, j \leq n\} \rangle$$

Hint: For $\mathcal{S} \subset G$,

$$\langle\langle \mathcal{S} \rangle\rangle = \{\, g_1 s_1^{n_1} g_1^{-1} \cdot \ldots \cdot g_k s_k^{n_k} g_k^{-1} \,|\, k \geq 0 \text{ and } g_i \in G, s_i \in \mathcal{S}, n_i \in \mathbb{Z} \;\forall\; i \leq k \}$$

(2) Show that $BS(2,3) \doteq \langle a, b \,|\, ba^2b^{-1} = a^3 \,\rangle$ (BS stands for "Baumslag–Solitar", its co-discoverers) is not *Hopfian*: there is an onto homomorphism $\phi\colon\ BS(2,3) \to BS(2,3)$ that is not an isomorphism.

Hint: You may assume that $a^{-1}bab^{-1}a^{-1}bab^{-1}a^{-1}$ is nontrivial in $BS(2,3)$ (extra credit for a nice proof of this fact).

(3) Classify finite graphs up to homotopy equivalence; ie, produce a list of finite graphs $\{X_\alpha\}$ (for α in an appropriate index set) with the property that for any finite graph X there is a unique α such that X_α is homotopy equivalent to X. (In particular, X_α is not homotopy equivalent to X_β for $\alpha \neq \beta$.)

(4) Hatcher, Section 1.3, Exercise 5.

(5) Hatcher, Section 1.3, Exercise 6.

Hint: The following fact might be useful: for any cover $p\colon\ \widetilde{X} \to X$, all but finitely many of the circles comprising X lift homeomorphically to $\widetilde{X}$. (Prove it if you use it.)