TOPOLOGY 2, HOMEWORK 1

(1) The exercises below are related to ones given in class.

(a) (1 pt) In a category $\mathcal{C}$, a morphism $f\colon X \to Y$ is *invertible* if there is a morphism $g\colon Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Show that a functor $F\colon \mathcal{C} \to \mathcal{D}$ takes invertible morphisms of $\mathcal{C}$ to invertible morphisms of $\mathcal{D}$.

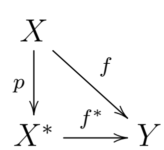
(b) (1 pt) Show that $\mathbb{R}P^n$ is Hausdorff. (Here $\mathbb{R}P^n = S^n/\sim$, where $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{y} = \pm\mathbf{x}$.)

(c) (1 pt) Give an example to show that a quotient projection $p\colon X \to X/\sim$, where $\sim$ is an equivalence relation on X, is not necessarily an open map. (Here a map $f\colon X \to Y$ is *open* if $f(U)$ is open in Y, for each open $U \subset X$.)

(2) Prove the following result:

Lemma. *Let X and Y be topological spaces, and suppose $X^* = X/\sim$ is a quotient space of X.*

(a) (1 pt) *A continuous map $f\colon X \to Y$ with the property that $f(x) = f(x')$ whenever $x \sim x'$ determines a unique continuous map $f^*\colon X^* \to Y$ so that the diagram below commutes.*



That is, $f = f^* \circ p$, where $p\colon X \to X^*$ is the quotient map.

(b) (1 pt) *If f^* is bijective, and f is either an open map or a closed map (ie, $f(C)$ is closed in Y for each closed $C \subset X$), then f^* is a homeomorphism.*

(c) (1 pt) *In particular, if X is compact, Y is Hausdorff, and f^* is bijective then f^* is a homeomorphism. (Hint: Show that this implies f is closed.)*

(3) (2 pts) Prove that S^n is homeomorphic to B^n/S^{n-1}, where $B^n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$.
Hint: Find a map $B^n \to S^n$ that allows you to use problem (2).

(4) (4 pts) For any $n \geq 1$, show that the three quotient spaces below are homeomorphic.

(a) $(\mathbb{R}^{n+1} - \{\mathbf{0}\})/\sim$, where $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{y} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}$.

(b) $\mathbb{R}P^n$; ie. $\mathbb{S}^n/\sim$, where $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{y} = \pm\mathbf{x}$.

(c) $B^n/\sim$, where $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{y} = \pm\mathbf{x}$ for $\mathbf{x} \in S^{n-1}$, and otherwise $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{y} = \mathbf{x}$.