TOPOLOGY 2, HOMEWORK 2

(1) The exercises below were given in class.

  (a) (1 pt) For an open set $U \subset S^{n-1}$ and $0 < \epsilon < 1$, show that the *radial ϵ-neighborhood of U*, defined as

$$\left\{ \mathbf{x} \;\middle|\; \|\mathbf{x}\| > 1 - \epsilon,\ \frac{\mathbf{x}}{\|\mathbf{x}\|} \in U \right\}$$

  is open in $\mathbb{D}^n$. (*Hint*: *Radial projection* $\mathbf{x} \mapsto \frac{\mathbf{x}}{\|\mathbf{x}\|}$ is continuous.)

  (b) Show that a CW complex with countably many cells is second-countable.

(2) (2 pts) For each $n \geq 0$, show that $\mathbb{R}P^n$ has a CW complex structure with a single k-cell for each $k \leq n$.

(3) Let $S^\infty = \{\mathbf{x} \in \mathbb{R}^\infty \mid \|\mathbf{x}\| = 1\}$, where

$$\mathbb{R}^\infty = \{(x_1, x_2, \ldots) \mid x_n = 0 \text{ for all but finitely many } n\}$$

is equipped with the norm $\|\mathbf{x}\| = \sqrt{\sum_{n=1}^{\infty}(x_i)^2}$. (Note that this sum is finite for each $\mathbf{x} \in \mathbb{R}^\infty$.) We equip $\mathbb{R}^\infty$ with the metric topology from this norm.

  (a) (1 pt) Show that S^∞ has a CW-complex structure with two k-cells for each $k \in \mathbb{N}$, when given the subspace topology from $\mathbb{R}^\infty$.

  (b) (1 pt) Show that $\mathbb{R}P^\infty = S^\infty/\mathbf{x} \sim -\mathbf{x}$ has the structure of a CW-complex with a single k-cell for each k, when given the quotient topology from S^∞.

(4) For any $g \geq 1$ fix a regular Euclidean $4g$-gon P_{4g}, and number its edges $e_0, \ldots, e_{4g-1}$ so that $e_i \cap e_{i-1}$ is a vertex v_i for each $i > 0$, and $e_0 \cap e_{4g-1}$ is a vertex v_0. Let Σ_g be the quotient space by the equivalence relation generated by the identifications below:

  - For $i = 0$ or 1 (mod 4), identify points of e_i with points of e_{i+2} by linearly extending $v_i \mapsto v_{i+3}$ and $v_{i+1} \mapsto v_{i+2}$.

Below I will write two relevant definitions and draw pictures of the identifications producing Σ_g, for $g = 1$ and 2. But first! Your assignment:

  (a) (2 pts) Show that Σ_g has a CW complex structure with one vertex.

  (b) (2 pts) Show that Σ_g is a 2-dimensional manifold.

Definition. For $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{y}_0$ and $\mathbf{y}_1 \in \mathbb{R}^n$, the map obtained by *linearly extending* $\mathbf{x}_0 \mapsto \mathbf{y}_0$ and $\mathbf{x}_1 \mapsto \mathbf{y}_1$ is given by $(1-t)\mathbf{x}_0 + t\mathbf{x}_1 \mapsto (1-t)\mathbf{y}_0 + t\mathbf{y}_1$. This maps the line segment joining $\mathbf{x}_0$ to $\mathbf{x}_1$ to the line segment joining $\mathbf{y}_0$ to $\mathbf{y}_1$.

Definition. For an arbitrary relation $\sim$ on a set X we define the equivalence relation $\simeq$ *generated by* $\sim$ by prescribing that $x \simeq y$ whenever $x \sim y$, and:

- $x \simeq x$ for all $x \in X$;
- $y \simeq x$ whenever $x \sim y$; and
- $x \simeq y$ whenever there is a sequence $x_0, x_1, \ldots, x_n$ such that $x_0 = x$, $x_n = y$, and either $x_i \sim x_{i-1}$ or $x_{i-1} \sim x_i$ for each $i > 0$.

In other words $\simeq$ is the "minimal" equivalence relation that includes $\sim$ as a subrelation.

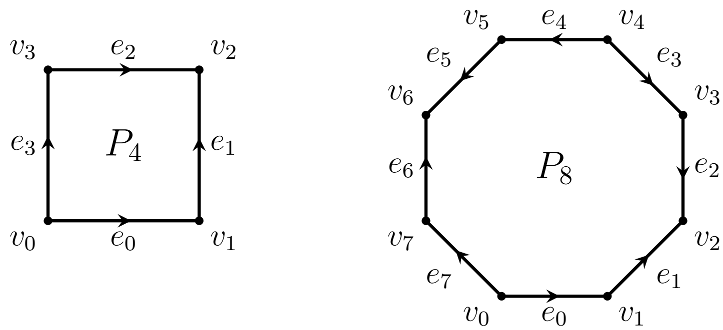

FIGURE 1. The polygons whose quotients are Σ_1 and Σ_2, respectively. Edges are identified respecting the orientations shown.