TOPOLOGY 2, HOMEWORK 5

(1) (a) (1 pt) Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 1$, and that $S^0 = \{\pm 1\}$ is not homeomorphic to S^n for any $n > 0$.
(b) (1 pt) If $f\colon X \to Y$ and $g\colon Y \to Z$ are homotopy equivalences, show that $g \circ f\colon X \to Z$ is a homotopy equivalence.

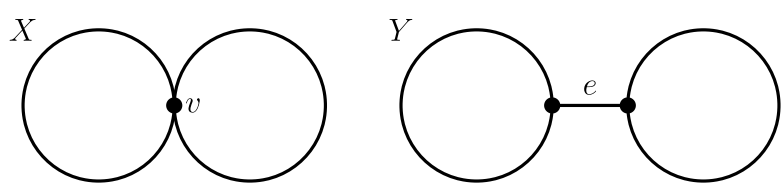
(2) (2 pts) Describe a CW complex structure on the quotient space $X = S_0^2 \sqcup S_1^2 / x_0 \sim x_1, y_0 \sim y_1$, where each S_i^2 is homeomorphic to the two-sphere and $x_i, y_i \in S_i^2$ are distinct points. In other words, X is obtained by identifying each of two distinct points in one copy of S^2 with each of two distinct points in another.

(3) (2 pts) Show that the pair (I, A) lacks the homotopy extension property, where

$$A = \{0\} \cup \left\{ \frac{1}{n} \,|\, n \in \mathbb{N} \right\}$$

(*Hint*: Suppose there is a retraction $I \times I \to (I \times \{0\}) \cup (A \times I)$, and think about what must happen on $I \times \{1\}$.)

(4) Consider the graphs X and Y pictured below (which you should regard as abstract CW complexes).



(a) (1 pt) Show that X does not embed into Y. (*Hint*: Find and use a neighborhood U of v in X such that $U - \{v\}$ has four components.)
(b) (2 pts) Explicitly describe a homotopy-inverse for the map $Y \to X$ that takes the edge e of Y and its endpoints to the vertex v of X and restricts on $Y - \bar{e}$ to a homeomorphism to $X - v$ (and show it is a homotopy-inverse).

(5) (2 pts) Hatcher Chapter 0, Exercise 20. (Draw a sequence of pictures for this one, and explain why subsequent spaces are homotopy equivalent.)