TOPOLOGY 2, THE *REAL* HOMEWORK 2

(1) The exercises below were given in class.

(a) For an open set $U \subset S^{n-1}$ and $0 < \epsilon < 1$, show that the *radial ϵ-neighborhood of U*, defined as

$$\left\{ \mathbf{x} \;\middle|\; \|\mathbf{x}\| > 1 - \epsilon,\ \frac{\mathbf{x}}{\|\mathbf{x}\|} \in U \right\}$$

is open in $\mathbb{D}^n$. (*Hint*: *Radial projection* $\mathbf{x} \mapsto \frac{\mathbf{x}}{\|\mathbf{x}\|}$ is continuous.)

(b) Show that a finite-dimensional CW complex with countably many cells is second-countable.

(2) Define:

- $\coprod \mathbb{R} = \{(x_1, x_2, \ldots) \,|\, x_i \in \mathbb{R} \text{ for each } i \in \mathbb{N}\}$
- $\ell^2 = \{(x_1, x_2, \ldots) \,|\, \sum_{i=1}^{\infty} (x_i)^2 < \infty\} \subset \coprod \mathbb{R}$
- $\mathbb{R}^\infty = \{(x_1, x_2, \ldots) \,|\, x_i = 0 \text{ for all but finitely many } i\} \subset \ell^2$

There is a norm on ℓ^2 given by $\|(x_1, x_2, \ldots)\| = \sqrt{\sum_{i=1}^{\infty} (x_i)^2}$, and this norm determines a metric on ℓ^2 via $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

(a) Define $bigS^\infty = \{\mathbf{x} \in \ell^2 \,|\, \|\mathbf{x}\| = 1\}$ and $lilS^\infty = \{\mathbf{x} \in \mathbb{R}^\infty \,|\, \|\mathbf{x}\| = 1\}$. Which of these is the union $S^\infty = \bigcup S^n$? (Here $S^n \subset \mathbb{R}^{n+1}$ is included in $\coprod \mathbb{R}$ by $\mathbf{x} \mapsto (\mathbf{x}, 0, \ldots)$ for each $n \geq 0$.)

(b) S^∞ inherits the structure of a CW complex as $\bigcup S^n$, where for each n, S^n is given a CW structure with two k-cells for each $k \leq n$. It can also take the product or box topology as a subspace of $\coprod \mathbb{R}$, or the metric topology as a subspace of ℓ^2. Which of these subspace topologies match the weak topology from its CW structure? Which do not?

(c) Show that $\mathbb{R}P^\infty = S^\infty / \mathbf{x} \sim -\mathbf{x}$ has the structure of a CW-complex with a single k-cell for each k, when given the quotient topology from S^∞.

(3) For any $g \geq 1$ fix a regular Euclidean $4g$-gon P_{4g}, and number its edges $e_0, \ldots, e_{4g-1}$ so that $e_i \cap e_{i-1}$ is a vertex v_i for each $i > 0$, and $e_0 \cap e_{4g-1}$ is a vertex v_0. Let Σ_g be the quotient space by the equivalence relation generated by the identifications below:

- For $i = 0$ or $1 \pmod 4$, identify points of e_i with points of e_{i+2} by linearly extending $v_i \mapsto v_{i+3}$ and $v_{i+1} \mapsto v_{i+2}$.

On the next page I will write two relevant definitions and draw pictures of the identifications producing Σ_g, for $g = 1$ and 2. But first! Your assignment:

(a) Show that Σ_g has a CW complex structure with one vertex.

(b) Show that Σ_g is a 2-dimensional manifold.

Definition. For $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{y}_0$ and $\mathbf{y}_1 \in \mathbb{R}^n$, the map obtained by *linearly extending* $\mathbf{x}_0 \mapsto \mathbf{y}_0$ and $\mathbf{x}_1 \mapsto \mathbf{y}_1$ is given by $(1-t)\mathbf{x}_0 + t\mathbf{x}_1 \mapsto (1-t)\mathbf{y}_0 + t\mathbf{y}_1$. This maps the line segment joining $\mathbf{x}_0$ to $\mathbf{x}_1$ to the line segment joining $\mathbf{y}_0$ to $\mathbf{y}_1$.

Definition. For an arbitrary relation $\sim$ on a set X we define the equivalence relation $\simeq$ *generated by* $\sim$ by prescribing that $x \simeq y$ whenever $x \sim y$, and:
- $x \simeq x$ for all $x \in X$;
- $y \simeq x$ whenever $x \sim y$; and
- $x \simeq y$ whenever there is a sequence $x_0, x_1, \ldots, x_n$ such that $x_0 = x$, $x_n = y$, and either $x_i \sim x_{i-1}$ or $x_{i-1} \sim x_i$ for each $i > 0$.

In other words $\simeq$ is the "minimal" equivalence relation that includes $\sim$ as a subrelation.

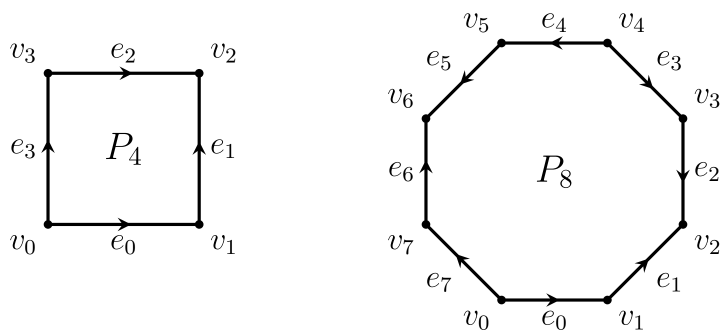

Figure 1. The polygons whose quotients are Σ_1 and Σ_2, respectively. Edges are identified respecting the orientations shown.