Marsden & Hoffman problems

1. Compute the second-order Taylor formula for $f(x,y) = e^x \cos y$ around $(0,0)$.
(That is, write $f(x,y) = f(0,0) + (Df)_{(0,0)}(x,y) + (D^2 f)_{(0,0)}((x,y),(x,y)) + R_2(x,y)$ in terms of only some explicit constants, x and y, and $R_2(x,y)$.)

2. Prove that $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ is negative definite if and only if $a < 0$ and $ad - b^2 > 0$.

PPotWs

August 2016, Problem 5. Suppose $f(x,y)$ is a C^2 function on $\mathbb{R}^2$ such that for some $M > 0$ and all (x,y) in the closed unit disk $\mathbb{D} = \{(x,y) \mid x^2 + y^2 \leq 1\}$,

$$[f_{xx}(x,y)]^2 + 2[f_{xy}(x,y)]^2 + [f_{yy}(x,y)]^2 \leq M.$$

If $f(0,0) = f_x(0,0) = f_y(0,0) = 0$, show that

$$\left| \iint_{\mathbb{D}} f(x,y)\, dx\, dy \right| \leq \frac{\pi\sqrt{M}}{4}$$

Remark: (At some point) use the formula for evaluating double integrals in polar coordinates:

$$\iint_{\mathbb{D}} g(x,y)\, dx\, dy = \int_0^{2\pi} \int_0^1 g(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

April 2012, Problem 5. For $a = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$, $a_n \neq 0$ let $P_a(x) = a_n x^n + \ldots + a_1 x + a_0$. Suppose that for $a^0 = (a_0^0, a_1^0, \ldots, a_n^0)$, $a_n^0 \neq 0$ the polynomial $P_{a^0}(x)$ has n distinct real roots. Prove that there exist $\epsilon > 0$ and C^∞ smooth functions

$$\lambda_1, \ldots, \lambda_n \colon B^{n+1}(a^0, \epsilon) \to \mathbb{R}$$

such that for any $a \in B^{n+1}(a^0, \epsilon)$, $\lambda_1(a), \ldots, \lambda_n(a)$ are distinct roots of the polynomial $P_a(x)$. In other words, prove that in a small neighborhood of a^0, roots of the polynomial P_a depend smoothly on the coefficients $a_0, a_1, \ldots, a_n$.