August 2015, Problem 6. Let $f \in C^1(\mathbb{R})$ be a continuously differentiable function such that $|f'(x)| \leq 1/2$ for all $x \in \mathbb{R}$. Define $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x, y) = (x + f(y), y + f(x)).$$

Prove that

(1) g is a diffeomorphism,
(2) $g(\mathbb{R}^2) = \mathbb{R}^2$,
(3) the area $|g([0,1]^2)|$ of the image of the unit square belongs to the interval $[3/4, 5/4]$.

Hint: Among other tools use the contraction principle.

April 2016, Problem 6. Suppose that smooth functions $f_k\colon \mathbb{R}^k \to \mathbb{R}$ are defined for $k = 1, 2, \ldots, 9$. Let $\Phi = (\phi_1, \ldots, \phi_{10})\colon \mathbb{R}^{10} \to \mathbb{R}^{10}$ be a mapping defined by

$$\begin{aligned}
\phi_1(x_1, \ldots, x_{10}) &= x_1 \\
\phi_2(x_1, \ldots, x_{10}) &= 2x_2 + f_1(x_1) \\
\phi_3(x_1, \ldots, x_{10}) &= 3x_3 + f_2(x_1, x_2) \\
&\cdots \\
\phi_{10}(x_1, \ldots, x_{10}) &= 10x_{10} + f_9(x_1, \ldots, x_9)
\end{aligned}$$

(1) Prove that Φ is a diffeomorphism of $\mathbb{R}^{10}$ onto an open subset of $\mathbb{R}^{10}$.
(2) Find the volume of $\Phi((-1,1)^{10})$.