Pugh Ch. 3, Problem 57. Construct a function $f\colon [-1,1] \to \mathbb{R}$ such that

$$\lim_{r\to 0}\left(\int_{-1}^{-r} f(x)\,dx + \int_{r}^{1} f(x)\,dx\right)$$

exists (and is a finite real number) but the improper integral $\int_{-1}^{1} f(x)\,dx$ does not exist. Do the same for a function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{R\to\infty}\int_{-R}^{R} g(x)\,dx$$

exists but the improper integral $\int_{-\infty}^{\infty} g(x)\,dx$ fails to exist. [Hint: The functions are not symmetric across 0.]

August 2011, Problem 1. Let $\beta > 0$, and let $\{u_n\}$ be a sequence of positive real numbers such that $\frac{u_{n+1}}{u_n} \le \beta$ for every $n \in \mathbb{N}$. Prove that

$$\limsup_{n\to\infty} \sqrt[n]{u_n} \le \limsup_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right).$$