Rudin, Ch. 7

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

3. Construct sequences $\{f_n\}$ and $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course $\{f_n g_n\}$ must converge on E).

6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

7. For $n = 1, 2, 3, \ldots$ and x real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

8. If

$$I(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 & \text{for } x > 0, \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \qquad (a \leq x \leq b)$$

converges uniformly, and that f is continuous at every $x \neq x_n$.

...and the **PPotW** (August 2016, Problem 3.)
For $n \geq 2$ define $f_n\colon [0,1] \to [0,1]$ by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq 1/n \\ \frac{n}{n-1}(1 - x) & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}$$

Show that $\sum_{n=2}^{\infty} [f_n(x)]^n$ converges pointwise on $[0, 1]$ to a function $f(x)$ that is continuous on $(0, 1]$, but that the improper integral $\int_0^1 f(x)\, dx$ diverges (the integral is improper at 0).