Spivak, Ch. 24

13. Prove that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is an even function, then $a_n = 0$ for n odd, and if f is an odd function then $a_n = 0$ for n even.

15. The *Fibonacci sequence* (a_n) is defined by $a_1 = a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$.

(a) Show that $a_{n+1}/a_n \leq 2$ for all n.

(b) Let

$$f(x) = \sum_{n=1}^{\infty} a_n x^{n-1} = 1 + x + 2x^2 + 3x^3 + \cdots$$

Use the ratio test to prove that $f(x)$ converges if $|x| < 1/2$.

(c) Prove that if $|x| < 1/2$ then

$$f(x) = \frac{-1}{x^2 + x - 1}.$$

Hint: This equation can be written $f(x) - xf(x) - x^2 f(x) = 1$.

(d) Use the partial fraction decomposition for $1/(x^2 + x - 1)$, and the power series for $1/(x - a)$, to obtain another power series for f.

(e) It follows from problem 12 that the two power series obtained for f must be the same. Use this fact to show that

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

18. Show that the series

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2}$$

converges uniformly to $\frac{1}{2}\log(x+1)$ on $[-a, a]$ for $0 < a < 1$, but that at 1 it converges to $\log 2$!

19. Suppose that $\sum_{n=0}^{\infty} c_n$ converges.

(a) Show that the power series $\sum c_n x^n$ converges uniformly on $[-a, a]$ for $0 < a < 1$.

(b) The power series may not converge at $x = -1$ (for example, the Taylor series of $f(x) = \log(1 + x)$). However, a theorem of Abel shows that the series *does* converge uniformly on $[0, 1]$. Consequently, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous on $[0, 1]$ and, in particular, $\sum_{n=0}^{\infty} a_n = \lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n$. Prove Abel's theorem by showing that if $|a_m + \ldots + a_n| < \epsilon$ then $|a_m x^m + \ldots + a_n x^n| < \epsilon$ using

Abel's Lemma. *For any $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,*

$$a_1 b_1 + \cdots + a_n b_n = s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + s_n b_n,$$

where $s_k = a_1 + \ldots + a_k$ for each k.

....and the **PPotW** (January 2002, **Problem 4.**)

Define the function

$$f(x) = \int_0^{x^2} e^{-t^2/2}\, dt$$

Show that $f(x)$ has a power series representation $\sum_{k=0}^{\infty} a_k x^k$ valid for all $x \in \mathbb{R}$, and compute the first two non-zero coefficients.