TOPOLOGY 2, HOMEWORK 1

(1) Show that $\mathbb{S}^0$ is not homeomorphic to $\mathbb{S}^n$ for any $n > 0$, and $\mathbb{S}^1$ is not homeomorphic to $\mathbb{S}^n$ for any $n > 1$.

(2) Let $\mathbb{R}^\omega = \{(x_1, x_2, \ldots) \,|\, x_i \in \mathbb{R} \;\forall\; i \in \mathbb{N}\}$, and for each $n \in \mathbb{N}$ define $f_n \colon \mathbb{S}^n \to \mathbb{R}^\omega$ by
$$f_n(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{n+1}, 0, 0, \ldots).$$
Define $\mathbb{S}^\infty = \bigcup_{i=1}^\infty f_n(\mathbb{S}^n)$. The *weak topology* on $\mathbb{S}^\infty$ (this is Hatcher's terminology) is defined as follows:
$$U \subset \mathbb{S}^\infty \text{ is open if and only if } f_n^{-1}(U) \text{ is open in } \mathbb{S}^n \text{ for all } n.$$
Show that the weak topology on $\mathbb{S}^\infty$ is at least as fine as the subspace topology from the box topology on $\mathbb{R}^\omega$ (i.e. that every open set in the latter topology is also open in the former), and strictly finer than the product topology.

(3) Show that with the weak topology, $\mathbb{S}^\infty$ is not second-countable.

(4) The *line with two origins* L is the quotient space of the disjoint union of two copies of $\mathbb{R}$ by setting each non-zero real number in the first copy equivalent to the same number in the second. Show that L has two of the three properties of a one-manifold (Hausdorff, second-countable, atlas of charts to $\mathbb{R}$), but not the third.

(5) Find a topological space X with an equivalence relation $\sim$ such that the quotient map $p\colon X \to X/\sim$ is not open. (Here a map $f\colon X \to Y$ is *open* if $f(U)$ is open in Y for each open $U \subset X$.)

(6) Prove that $\mathbb{S}^n$ is homeomorphic to $D^n/\mathbb{S}^{n-1}$, where D^n is the closed unit disk in $\mathbb{R}^n$; i.e. $D^n = \{\mathbf{x} \in \mathbb{R}^n \,|\, \|\mathbf{x}\| \leq 1\}$.