TOPOLOGY 2, HOMEWORK 2

(1) Prove the following result:

Proposition. *For a continuous map $f\colon X \to Y$ of topological spaces, and a quotient space $X^* = X/\sim$ of X, show that if $f(x) = f(x')$ whenever $x \sim x'$ then there is a unique continuous map $f^*\colon X^* \to Y$ so that $f = f^* \circ p$, where $p\colon X \to X^*$ is the quotient map. That is, the diagram below commutes.*

$$\begin{array}{ccc} X & & \\ {\scriptstyle p}\downarrow & \searrow^{f} & \\ X^* & \xrightarrow{f^*} & Y \end{array}$$

Moreover, show that if f^* is bijective, and f is either an open map or a closed map (ie, $f(C)$ is closed in Y for each closed $C \subset X$), then f^* is a homeomorphism.

(2) (a) For $f\colon \mathbb{D}^n \times \{0, 1\} \to \mathbb{S}^n$ defined by

$$f(\mathbf{x}, 0) = (\mathbf{x}, -\sqrt{1 - \|\mathbf{x}\|^2}) \quad \text{and} \quad f(\mathbf{x}, 1) = (\mathbf{x}, \sqrt{1 - \|\mathbf{x}\|^2}),$$

show that the induced map f^* on $\mathbb{D} \times \{0, 1\}/\sim$ is a bijection, where $(\mathbf{x}, i) \sim (\mathbf{x}', j)$ if and only if either $\mathbf{x} = \mathbf{x}'$ and $j = i$ or $\mathbf{x} = \mathbf{x}' \in \mathbb{S}^{n-1}$ and $j = 1 - i$.

(b) For any two pairs of distinct points $\mathbf{x} \neq \mathbf{y} \in \mathbb{S}^n$ and $\mathbf{x}' \neq \mathbf{y}' \in \mathbb{S}^n$, show that there is a homeomorphism $h\colon \mathbb{S}^n \to \mathbb{S}^n$ with $h(\mathbf{x}) = \mathbf{x}'$ and $h(\mathbf{y}) = \mathbf{y}'$.

(3) Show for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and each $i \in \{1, \ldots, n+1\}$ that the map $\mathbb{K}^n \to \mathbb{K}P^n$ defined below is a homeomorphism to an open subset of $\mathbb{K}P^n$:

$$(x_1, \ldots, x_n) \mapsto [x_1 : \ldots : x_{i-1} : 1 : x_i : \ldots : x_n].$$

Here $[y_1 : \ldots : y_{n+1}]$ refers to the equivalence class of $(y_1, \ldots, y_{n+1}) \in \mathbb{K}^{n+1} - \{\mathbf{0}\}$ in $\mathbb{K}P^n$.

(4) Show that $\mathbb{C}P^n \approx \mathbb{S}^{2n+1}/\sim$ where $\mathbf{z} \sim \mathbf{w}$ if and only if $\mathbf{z} = \lambda\mathbf{w}$ for some $\lambda \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

(5) Show that $\mathbb{R}P^1 \approx \mathbb{S}^1$, and $\mathbb{C}P^1 \approx \mathbb{S}^2$. (*Hint*: the first two are each homeomorphic to the one-point compactification of $\mathbb{R}$, and the latter two to that of $\mathbb{R}^2$.)

(6) Show that $\mathbb{R}P^n$ is homeomorphic to the quotient space of $\mathbb{D}^n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$ by setting $\mathbf{x} \sim -\mathbf{x}$ for each $\mathbf{x} \in \mathbb{S}^{n-1}$.