Final Exam

Each problem is worth 10 points. Do four of five.

You may use facts proved in Chapters 1 – 5 of the book or in class. Prove anything else that you use, unless it is explicitly indicated otherwise.

(1) (August 2016, Problem 2) Let V be a finite dimensional vector space and $A\colon V \to V$ a linear map. Suppose $Null(A) = Null(A^2)$. Show that for any integer $m > 0$ we have $Null(A) = Null(A^m)$.

(*Hint*: How does the range of A intersect its null space?)

(2) For vectors $\alpha = (a_1, a_2, a_3)$ and $\beta = (b_1, b_2, b_3)$ in $\mathbb{R}^3$, the *cross product* of α with β is defined as

$$\alpha \times \beta = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

(a) For a fixed arbitrary vector α, $T_\alpha(\beta) = \alpha \times \beta$ defines a linear transformation $T_\alpha\colon \mathbb{R}^3 \to \mathbb{R}^3$. Find the matrix M_α of T_α relative to the standard basis.

(b) For $\alpha = (1, 2, 2)$, find an invertible matrix P with the property that

$$P^{-1}M_\alpha P = \begin{pmatrix} 0 & -3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(You may take for granted that $\alpha \times \beta$ is *orthogonal* to α and β; that is, regarding vectors as row matrices in $\mathbb{R}^{1\times n}$, that $(\alpha \times \beta)\alpha^t = \mathbf{0} = (\alpha \times \beta)\beta^t$.)

(3) (a) For a fixed non-zero column matrix $X \in \mathbb{R}^{n\times 1}$ let $M = XX^t \in \mathbb{R}^{n\times n}$. What is the rank of M? What is $det(M)$?

(b) Show that $M^2 = \|X\|^2 M$, where $\|X\|^2 = X^tX$. Use this to identify (with proof) the monic generator for the ideal $\{f \mid f(M) = 0\} \subset \mathbb{R}[x]$.

(4) Let $V_n \subset \mathbb{R}[x]$ be the space of polynomials of degree at most n.

(a) For any $a < b \in \mathbb{R}$ and distinct $t_0, \ldots, t_n \in \mathbb{R}$, show that there exist $m_0, \ldots, m_n \in \mathbb{R}$ such that for all $p \in V_n$,

$$\int_a^b p(x)\, dx = m_0p(t_0) + \ldots + m_np(t_n)$$

(You may assume that the definite integral defines a linear function $V_n \to \mathbb{R}$.)

(b) For $[a, b] = [0, 1]$ and $t_0 = -1$, $t_1 = 1$, $t_2 = 2$, identify m_0, m_1, and m_2 such that the above holds for all $p \in V_2$.

(5) (August 2016, Problem 3) Let $\{v_1, v_2, v_3\} \subset \mathbb{Z}^3$ be vectors with integer coordinates. Show that every vector in $\mathbb{Z}^3$ can be expressed as a linear combination of the v_i with integer coefficients if and only if the (Euclidean 3-dimensional) volume of the parallelepiped they form is equal to 1. The parallelepiped formed by the v_i is:

$$P = \{c_1v_1 + c_2v_2 + c_3v_3 \mid 0 \le c_i \le 1 \ \forall\ i = 1, 2, 3\}.$$