S^∞ IS NOT SECOND-COUNTABLE

We will assume that $S^\infty = \bigcup S^n$, topologized with the weak topology, has a countable basis $\mathcal{B} = \{U_n\}_{n\in\mathbb{N}}$ of open sets, and produce a contradiction using a diagonal argument. Pass to the subset of $\mathcal{B}$ consisting of the U_n which contain $(1, 0, 0, \ldots)$, and re-number this set as $\{U_n\}$. Below we will produce an open set V containing $(1, 0, 0, \ldots)$ that contains no U_n. Thus $\mathcal{B}$ is not a basis, a contradiction.

For each $n > 0$ let $\epsilon_n > 0$ be such that $U_n \cap S^n$ contains a ball of radius ϵ_n around $(1, 0, \ldots, 0)$. (Here "ball of radius ϵ_n" refers to the intersection with S^n of such a ball in the standard metric on $\mathbb{R}^{n+1}$.) Then let

$$V = S^\infty \cap \left((0, 2) \times \prod_{n=1}^{\infty} (-\epsilon_n/2, \epsilon_n/2) \right)$$

V is open in S^∞ since for any n, $V \cap S^n$ is the intersection with S^n of:

$$(0, 2) \times (-\epsilon_1/2, \epsilon_1/2) \times \cdots \times (-\epsilon_n/2, \epsilon_n/2),$$

which is open in $\mathbb{R}^{n+1}$. We claim now that for any n, $U_n \cap S^n$ is not contained in $V \cap S^n$. Consider the curve γ_n in S^n defined by $\gamma_n(t) = (\cos t, 0, \ldots, 0, \sin t)$. It exits $V \cap S^n$ before $U_n \cap S^n$: in particular, for t_0 such that $\sin t_0 = \epsilon_n/2$ we have

$$\|\gamma_n(t_0) - (1, 0, \ldots, 0)\| = \sqrt{2(1 - \cos t_0)} = \frac{\epsilon_n}{2} \frac{\sqrt{2}}{\sqrt{1 + \cos t_0}} < \frac{\epsilon_n}{\sqrt{2}}$$

And Bob, as they say, is your uncle.