# THE TENSOR CHARACTERIZATION LEMMA

**Lemma.** *For a smooth manifold $M$ let $\mathfrak{X}(M)$ denote the set of smooth vector fields on $M$ and $C^\infty(M)$ the set of smooth functions $M \to \mathbb{R}$. A function*

$$A\colon \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{s} \to C^\infty(M) \quad \textit{or} \quad \mathfrak{X}(M)$$

*determines a $(0,s)$- or $(1,s)$-tensor field on $M$, respectively, if and only if $A$ is multilinear over $\mathbb{R}$ and for any $f_1, \ldots, f_s \in C^\infty(M)$ and $\mathbf{X}_1, \ldots, \mathbf{X}_s \in \mathfrak{X}(M)$,*

$$A(f_1\mathbf{X}_1, \ldots, f_s\mathbf{X}_s)(p) = f_1(p) \cdots f_s(p)\, A(\mathbf{X}_1, \ldots, \mathbf{X}_s)(p) \tag{1}$$

*for all $p \in M$.*

*Proof.* Any $(0,s)$- or $(1,s)$-tensor field $A$ on $M$ defines a map on vector fields by:

$$A(\mathbf{X}_1, \ldots, \mathbf{X}_s)(p) = A_p(\mathbf{X}_1|_p, \ldots, \mathbf{X}_s|_p) \quad \text{for all } p \in M$$

It is straighforward to check in local coordinates that the function or vector field so-defined is smooth.

Now suppose that $A$ is a function on $\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)$ which is multilinear over $\mathbb{R}$ and has property (1) above. I'll prove in the case that $A$ maps to $\mathfrak{X}(M)$ that it determines a unique $(1,s)$-tensor field. The following fact is key:

**Fact.** For some $p \in M$, if $\mathbf{X}_i \equiv \mathbf{Y}_i$ for each $i$ on a neighborhood $U$ of $p$ in $M$ then $A(\mathbf{X}_1, \ldots, \mathbf{X}_s)(p) = A(\mathbf{Y}_1, \ldots, \mathbf{Y}_s)(p)$.

*Proof of Fact.* Let $f \in C^\infty(M)$ satisfy $\operatorname{supp}(f) \subset U$ and $f(q) = 1$ for all $q$ in an open set $U_0 \subset U$ with $p \in U_0$. (Such a function $f$ is called a *bump function*; you can prove it exists using partitions of unity.) The sequence of equations below proves the fact:

$$\begin{aligned}
&A(f\mathbf{X}_1, \ldots, f\mathbf{X}_s)(p) = f(p)^s A(\mathbf{X}_1, \ldots, \mathbf{X}_s)(p) = A(\mathbf{X}_1, \ldots, \mathbf{X}_s)(p) \\
&A(f\mathbf{Y}_1, \ldots, f\mathbf{Y}_s)(p) = f(p)^s A(\mathbf{Y}_1, \ldots, \mathbf{Y}_s)(p) = A(\mathbf{Y}_1, \ldots, \mathbf{Y}_s)(p) \\
&A(f\mathbf{X}_1, \ldots, f\mathbf{X}_s) \equiv A(f\mathbf{Y}_1, \ldots, f\mathbf{Y}_s)
\end{aligned}$$

The last equation holds because $f\mathbf{X}_i$ is identical to $f\mathbf{Y}_i$ on all of $M$ for each $i$ — they agree on $U$ since $\mathbf{X}_i$ and $\mathbf{Y}_i$ agree there, and outside $U$ each is identically $\mathbf{0}$. □

Now fix a chart $\phi = (x^1, \ldots, x^n)\colon U \to \mathbb{R}^n$ for an open set $U \subset M$, and for $p \in U$ let $f$ be a bump function with the properties listed in the proof of the Fact. For $q \in U_0$ and $i, j_1, \ldots, j_s \in \{1, \ldots, n\}$ define $A^i_{j_1,\ldots,j_s}(q)$ by

$$A\left(f\frac{\partial}{\partial x^{j_1}}, \ldots, f\frac{\partial}{\partial x^{j_s}}\right)(q) = \sum_{i=1}^n A^i_{j_1,\ldots,j_s}(q)\, \frac{\partial}{\partial x^i}\bigg|_q$$

The reason we are multiplying the $\frac{\partial}{\partial x^{j_i}}$'s by $f$ here is to produce smooth vector fields on all of $M$ (which are $\mathbf{0}$ outside $U$). The functions $A^i_{j_1,\ldots,j_s}$ so-defined on $U_0$ are smooth since $A$ sends $s$-tuples of smooth vector fields to smooth vector fields. Applying this procedure at each $p \in U$ yields well-defined (by the Fact) functions

$A^i_{j_1,\dots,j_s}$ on all of $U$. Moreover, for any smooth vector fields $\mathbf{X}_1, \dots, \mathbf{X}_s$ on $M$ with $\mathbf{X}_i$ given in local coordinates by $\mathbf{X}_i = \sum_{j=1}^n \xi^j_{(i)} \frac{\partial}{\partial x^j}$, for any $p \in U$ we have:

$$\begin{aligned} A(\mathbf{X}_1, \dots, \mathbf{X}_s)(p) &= A(f\mathbf{X}_1, \dots, f\mathbf{X}_s)(p) \\ &= \sum_{j_1,\dots,j_s} \xi^{j_1}_{(1)}(p) \cdots \xi^{j_s}_{(s)}(p) A\left(f\frac{\partial}{\partial x^{j_1}}, \dots, f\frac{\partial}{\partial x^{j_s}}\right)(p) \end{aligned}$$

The former equation follows from the Fact, the latter from multilinearity of $A$. It now follows from the definition of the $A^i_{j_1,\dots,j_s}$ that in local coordinates on $U$,

$$A = \sum_{i,j_1,\dots,j_s} A^i_{j_1,\dots,j_s} \frac{\partial}{\partial x^i} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$$

More precisely: the formula on the right defines a tensor $A_U$ on $U$ which is unique with the property that for any smooth vector fields $\mathbf{X}_1, \dots, \mathbf{X}_s$ on $M$,

$$A_U(\mathbf{X}_1|_U, \dots, \mathbf{X}_s|_U) = A(\mathbf{X}_1, \dots, \mathbf{X}_x)|_U$$

Here uniqueness stems from the fact that the coefficient functions $A^i_{j_1,\dots,j_s}$ are uniquely determned. The fact that $A_U$ is uniquely determined by $A$ ensures that the tensor $A_V$ analogously determined by $A$ on a different chart domain $V$ agrees with $A_U$ on the overlap $U \cap V$, hence that the local tensors $A_U$ patch together to determine a well-defined tensor on $M$. $\square$