SOME TRIGONOMETRIC FORMULAS FOR PARTIALLY TRUNCATED HYPERBOLIC TRIANGLES AND TETRAHEDRA

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As anyone knows who has glanced through Chapter VI of Fenchel’s *Elementary geometry in hyperbolic space* [3], there are multiple hyperbolic versions of the laws of sines and cosines. The extra ones are obtained by regarding various hyperbolic \( n \)-gons with some right angles as “triangles” that have some vertices outside the hyperbolic plane, for \( n = 4, 5 \) and \( 6 \). The hyperboloid model, in which \( \mathbb{H}^2 \) sits as a subspace of \( \mathbb{R}^3 \) which has been equipped with a certain non positive-definite bilinear form—the Lorentzian inner product—supplies a useful perspective on this. Here, vectors of the ambient \( \mathbb{R}^3 \) carry information about different objects of \( \mathbb{H}^2 \), depending on the sign of their self-pairing, and we can leverage this to efficiently encode any such object using just three vectors.

Similarly, “truncated tetrahedra” in \( \mathbb{H}^3 \), which are homeomorphic to affine simplices with certain vertices or their open neighborhoods removed, are encoded by four-tuples of vectors in an ambient Lorentzian \( \mathbb{R}^4 \). This perspective is used to prove trigonometric formulas in Chapter 3 of Ratcliffe’s *Foundations of hyperbolic manifolds* [4]. We follow it in this note to prove a few more which are not found in [3] nor in [4].

1. Background: the meaning of vectors in the hyperboloid model

We will follow the notation of [4, Ch. 3], which we now review, in describing the hyperboloid model of hyperbolic space. The Lorentzian inner product of \( \mathbf{x} = (x_1, \ldots, x_n) \) and \( \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n \) is defined as

\[
\mathbf{x} \circ \mathbf{y} = -x_1 y_1 + x_2 y_2 + \ldots + x_n y_n,
\]

and \( \mathbf{x} \) is said to be space-like, light-like, or time-like respectively as \( \mathbf{x} \circ \mathbf{x} \) is positive, zero, or negative. The Lorentzian norm of \( \mathbf{x} \) is \( \|\mathbf{x}\| = \sqrt{\mathbf{x} \circ \mathbf{x}} \), where the square root is taken to be positive, zero, or positive imaginary in the respective cases above. The light cone is the set of light-like vectors, and its interior is the set of time-like vectors. A time-like or light-like vector is positive if its first entry is. The hyperboloid model \( \mathbb{H}^{n-1} \) of hyperbolic space is the set of positive vectors with Lorentzian norm \( i \) in \( \mathbb{R}^n \), equipped with the distance \( d_H \) defined by

\[
cosh d_H(\mathbf{u}, \mathbf{v}) = -\mathbf{u} \circ \mathbf{v}.
\]

(We note that the following version of the Cauchy-Schwartz inequality follows from the usual one: for positive vectors \( \mathbf{x} \) and \( \mathbf{y} \) with \( \mathbf{x} \circ \mathbf{x} \leq 0 \) and \( \mathbf{y} \circ \mathbf{y} \leq 0 \), \( \mathbf{x} \circ \mathbf{y} \leq -\sqrt{(\mathbf{x} \circ \mathbf{x})(\mathbf{y} \circ \mathbf{y})} \), with equality if and only if they are linearly dependent, see eg. formula (1.0.2) of [1].)

The distance function \( d_H \) above is determined by the Riemannian metric on \( \mathbb{H}^{n-1} \) given by restricting the Lorentzian inner product to \( T_u \mathbb{H}^{n-1} = \{\mathbf{v} \mid \mathbf{v} \circ \mathbf{u} = 0\} \). (This restriction is positive-definite since \( \mathbf{u} \) is time-like, see [4, Theorem 3.1.4].) In particular, given \( \mathbf{x} \in \mathbb{H}^n \) and \( \mathbf{y} \in T_x \mathbb{H}^n \) with \( \mathbf{y} \circ \mathbf{y} = 1 \), it is easy to check that \( \gamma(t) = \cosh t \mathbf{x} + \sinh t \mathbf{y} \) is a geodesic in \( \mathbb{H}^n \) with \( \gamma(0) = \mathbf{x} \) and \( \gamma'(0) = \mathbf{y} \).

The most useful feature of the hyperboloid model for us is that vectors of \( \mathbb{R}^{n+1} \) which are not time-like encode certain codimension-one geometric objects in \( \mathbb{H}^n \). Here is the first:
Definition 1.1. Each positive light-like vector \( x \in \mathbb{R}^{n+1} \) determines a horosphere \( S = \{ v \in \mathbb{H}^n | v \circ x = -1 \} \). The horoball bounded by \( S \) is the set \( B = \{ v \in \mathbb{H}^n | v \circ x \geq -1 \} \).

A little multivariable calculus shows that the horoball \( S \) determined by a positive light-like vector \( x \in \mathbb{R}^{n+1} \) is the smooth submanifold \( f^{-1}(-1) \) of \( \mathbb{H}^n \), where \( f(u) = u \circ x \), and its tangent space at any \( u_0 \in S \) is \( T_{u_0}S = \{ v \in \mathbb{R}^{n+1} | v \circ u_0 = 0 = v \circ x \} \). For any such \( u_0 \) one may check directly that the formula \( F(v) = u_0 + v + \left( \frac{v \circ x}{2} \right) x \) defines a Riemannian isometry from \( T_{u_0}S \), equipped with the restriction of the Lorentzian inner product, to \( S \subset \mathbb{H}^n \). Since the inner product’s restriction is positive-definite on \( T_{u_0}S \), this explicitly confirms the well known fact that \( S \) is an isometrically embedded copy of the Euclidean space \( \mathbb{R}^{n-1} \). It also yields the following formula for the Euclidean distance \( d_S(u_0, u_1) \) in \( S \) between vectors \( u_0 \) and \( u_1 \):

\[
d_S(u_0, u_1) = \sqrt{-2(1 + u_0 \circ u_1)}
\]

To see this, set \( F(v) = u_1 \) and solve for \( v \circ v \) by taking the Lorentzian inner product of both sides with \( u_0 \). Using the formula for \( d_H(u_0, u_1) \) given above we obtain the comparison equation \( d_S(u_0, u_1)/2 = \sinh(d_H(u_0, u_1)/2) \). We note that this implies in particular that the isometric embedding \( F \) is proper; that is, \( S \) has compact intersection with any compact set of \( \mathbb{H}^n \).

Lemma 1.2. For \( v \in \mathbb{H}^n \) and a positive light-like vector \( x \), the signed hyperbolic distance \( d \) from \( v \) to the horosphere \( S \) determined by \( x \) satisfies \( e^d = -v \circ x \), where the sign of \( d \) is positive if \( v \) lies outside the horoball \( B \) bounded by \( S \). This distance is realized at \( t = d \) on

\[
\gamma(t) = e^{-t}v - \frac{\sinh t}{x \circ v} x,
\]

which is a parametrized geodesic through \( v \) in the direction of \( x \).

Proof. A vector \( u \in \mathbb{R}^{n+1} \) lies in \( S \) if and only if \( u \circ u = -1 \), so it lies in \( \mathbb{H}^n \), and \( u \circ x = -1 \). By the theory of Lagrange multipliers, the restriction of \( f(u) = u \circ v \) to \( S \) may attain a local extremum at \( u \in S \) only if the gradient of \( f \) at \( u \) is a linear combination of the gradients of the constraint functions \( g_1(u) = u \circ x \) and \( g_2(u) = u \circ u \). By a direct computation, \( \nabla f(u) = v \), \( \nabla g_1(u) = x \), and \( \nabla g_2(u) = u \), where \( v \) is obtained from \( v \) by switching the sign of first entry, and similarly for the others. It follows that at any local extremum of the restriction of \( f \) to \( S \), \( v \) is a linear combination of \( x \) and \( u \).

Since \( v \), which is time-like, is not a multiple of \( x \), which is light-like, this implies that we can express \( u \) in terms of \( v \) and \( x \). Upon plugging \( u = ax + bv \) into the constraints and solving for \( a, b \in \mathbb{R} \) we obtain the unique solution

\[
(1) \quad u = \frac{1}{2} \left( 1 - \frac{1}{(v \circ x)^2} \right) x - \frac{1}{v \circ x} v.
\]

The value of \( f \) at \( u \) is thus \( u \circ v = \frac{1}{2} \left( (v \circ x) + \frac{1}{v \circ x} \right) \), so by the definition of the hyperbolic distance \( d_H \) we have

\[
cosh d_H(u, v) = \frac{1}{2} \left( -v \circ x + \frac{1}{-v \circ x} \right).
\]

Therefore \( e^{d_H(u, v)} \) is either \( -v \circ x \) or its reciprocal, whichever is at least 1 since \( d_H(u, v) \) is non-negative. If we take \( d \) to be the signed distance, with non-negative sign if \( v \) is outside the horoball \( B \), then by the definition of \( B \) we have \( e^d = -v \circ x \) in all cases.

We finally note that \( d \) really is the (signed) distance from \( v \) to \( S \); that is, the unique critical point \( u \) of \( f \) described above is the global maximizer for the values of \( f \) on \( S \), so \( d_H(x, u) \) is the global minimizer of distances from \( v \) to points of \( S \). This follows from uniqueness and the
fact that as \( u \in \mathbb{H}^n \) escapes compact sets, \( f(u) \to -\infty \). Toward the latter point, note for an arbitrary \( u = (u_1, \ldots, u_{n+1}) \in \mathbb{H}^n \) that \( u_1 = \sqrt{1 + u_2^2 + \ldots + u_{n+1}^2} \), so we can rewrite \( f(u) \) as

\[
f(u) = -\sqrt{(1 + u_2^2 + \ldots + u_{n+1}^2)(1 + v_2^2 + \ldots + v_{n+1}^2) + u_2 v_2 + \ldots + u_{n+1} v_{n+1}}
\]

In passing from the first to the second line above we use the fact that \( \sqrt{a - \sqrt{b}} = (a - b)/(\sqrt{a} + \sqrt{b}) \). Expanding the numerator, canceling certain terms, and rearranging yields:

\[
-1 - (u_2^2 + \ldots + u_{n+1}^2) - (v_2^2 + \ldots + v_{n+1}^2) - \sum_{i \neq j} (u_i - v_j)^2.
\]

The denominator is at most some fixed multiple of \( \sqrt{1 + u_2^2 + \ldots + u_{n+1}^2} \), by the Cauchy-Schwarz inequality, whereas the numerator is at most the opposite of the square of this quantity. So as claimed, \( f(u) \to -\infty \) as \( u \) escapes compact sets.

For the parametrized curve \( \gamma \) defined in the statement, direct computation reveals that \( \gamma(t) \circ \gamma(t) = -1 \) for all \( t \), so \( \gamma \) maps into \( \mathbb{H}^n \), and that \( \gamma''(t) = \gamma(t) \). Therefore \( \gamma \) is a hyperbolic geodesic, by [4, Theorem 3.2.4]. More direct computation shows that \( \gamma(0) = v \) and \( \gamma(d) \) is the nearest point \( u \) to \( v \) on \( S \) described in (1). \( \square \)

**Lemma 1.3.** For linearly independent positive light-like vectors \( x_0 \) and \( x_1 \) of \( \mathbb{R}^{n+1} \), the minimum signed distance \( d \) from points on \( S_1 \) to \( S_0 \) satisfies \( e^d = -\frac{1}{2} x_0 \circ x_1 \), where \( S_i \) is the horosphere of \( \mathbb{H}^n \) determined by \( x_i \) for \( i = 0, 1 \). This distance is uniquely attained by points at \( t = \pm d/2 \) on the geodesic

\[
\gamma(t) = \frac{1}{\sqrt{-2(x_0 \circ x_1)}} (e^t x_0 + e^{-t} x_1)
\]

from \( x_1 \) to \( x_0 \).

**Proof.** A vector \( u \in \mathbb{R}^{n+1} \) lies in \( S_1 \) if and only if \( u \circ u = -1 \), \( u \) is positive, and \( u \circ x_1 = -1 \).

By the theory of Lagrange multipliers, the restriction of \( f(u) = u \circ x_0 \) to \( B_1 \) may attain a local extremum at \( u \in S_1 \) only if the gradient of \( f \) at \( u \) is a linear combination of the constraint gradients \( \nabla g_1(u) \) and \( \nabla g_2(u) \), where \( g_1(u) = u \circ x_1 \) and \( g_2(u) = u \circ u \).

Direct computation yields \( \nabla f(u) = x_0 \), \( \nabla g_1(u) = x_1 \), and \( \nabla g_2(u) = 2u \), where \( x_0 \) is obtained from \( x_0 \) by multiplying the first entry by \(-1\) and similarly for the others. It thus follows that at such a local extremum \( u \), \( x_0 \) is a linear combination of \( x_1 \) and \( u \) so, since \( x_0 \) is not a multiple of \( x_1 \), \( u \) is a linear combination of the \( x_i \).

Plugging \( u = ax_0 + bx_1 \) into the constraint equations and solving for \( a, b \in \mathbb{R} \) yields

\[
u = -\frac{1}{x_0 \circ x_1} x_0 + \frac{1}{2} x_1.
\]

This is a positive vector since it is a positive linear combination of the positive vectors \( x_0 \) and \( x_1 \). By Lemma 1.2 and a direct computation, the signed distance \( d \) from \( u \) to \( S_0 \) satisfies \( e^d = -\frac{1}{2} x_0 \circ x_1 \).

Substituting \( u \) for \( v \) in the formula for the geodesic \( \gamma(t) \) defined in Lemma 1.2 and simplifying yields

\[
\gamma(t) = \frac{e^t}{-x_0 \circ x_1} x_0 + \frac{e^{-t}}{2} x_1.
\]

Note that \( \gamma(0) = u \in S_1 \) and \( \gamma(d) \in S_0 \). The more-symmetric formula given in the statement is obtained by translating the parametrization, replacing \( t \) by \( t - d/2 \).
It remains to show for $\mathbf{u}$ from the formula (2) that $f(\mathbf{u})$ is a global maximum of $f$ on $S_1$, hence that $d$ is a global minimum of the signed distance to $S_0$ on $S_1$. This follows from the fact that $\mathbf{u}$ is the unique critical point of $f$ on $S_1$, together with the fact that $f(\mathbf{v}) \rightarrow -\infty$ as $\mathbf{v} \in S_1$ escapes compact sets. Indeed, for any fixed $r < 0$, and any $\mathbf{v} \in S_1$ such that $f(\mathbf{v}) \geq r$, we have $\mathbf{v} \circ \mathbf{u} = -f(\mathbf{v})/\mathbf{x}_0 \circ \mathbf{x}_1 - 1/2 \geq -r/\mathbf{x}_0 \circ \mathbf{x}_1 - 1/2$, so $\mathbf{v}$ is contained in the closed ball of radius $\cosh^{-1}(r/\mathbf{x}_0 \circ \mathbf{x}_1 + 1/2)$ around $\mathbf{u}$. This ball is compact. \hfill □

The set of totally geodesic subspaces of $\mathbb{H}^n$ with dimension $k$ corresponds one-to-one with the set of $(k + 1)$-dimensional vector subspaces of $\mathbb{R}^{n+1}$ that intersect $\mathbb{H}^n$, via $V \mapsto V \cap \mathbb{H}^n$. In particular, for $x \in \mathbb{H}^n$ and $y \in T_x \mathbb{H}^n$, the geodesic described above parametrizes the intersection with $\mathbb{H}^n$ of the two-dimensional subspace $\text{Span}\{x, y\} \subset \mathbb{R}^{n+1}$.

Conversely, a hyperplane $P$ of $\mathbb{H}^n$ — that is, a totally geodesic subspace of codimension 1 — is contained in a codimension-1 subspace $V$ of $\mathbb{R}^{n+1}$. There is thus a 1-dimensional subspace of space-like vectors $\mathbf{y} \in \mathbb{H}^{n+1}$ with the property that $\mathbf{x} \circ \mathbf{y} = 0$ for all $\mathbf{x}$ in $V$. This motivates:

**Definition 1.4.** For a space-like vector $\mathbf{y}$, the polar hyperplane to $\mathbf{y}$ is $P = \{x \in \mathbb{H}^n | x \circ \mathbf{y} = 0\}$. The half-space bounded by $P$ with outward normal $\mathbf{y}$ is $H = \{x \in \mathbb{H}^n | x \circ \mathbf{y} \leq 0\}$.

This observation motivates a series of important geometric interpretations on the Lorentz pairing between vectors of various types.

**Lemma 1.5.** Suppose $\mathbf{x} \in \mathbb{R}^{n+1}$ is a positive light-like vector and $H \subset \mathbb{H}^n$ is a hyperplane with ideal boundary disjoint from $\mathbf{x}$, and let $\mathbf{y} \in \mathbb{R}^{n+1}$ be the unit space-like vector such that $\mathbf{u} \circ \mathbf{y} = 0$ for all $\mathbf{u} \in H$ and $\mathbf{y} \circ \mathbf{x} < 0$. Then $\mathbf{y}$ and $\mathbf{x}$ are on opposite sides of the codimension-one subspace of $\mathbb{R}^{n+1}$ containing $H$, and the minimal signed distance $h$ from $H$ to $B$ satisfies $e^h = -\mathbf{x} \circ \mathbf{y}$. It is realized along a unique geodesic arc perpendicular to $H$ in the direction of $B$.

**Proof.** A vector $\mathbf{v} \in \mathbb{R}^{n+1}$ lies in $H$ if and only if $\mathbf{v} \circ \mathbf{x} = -1$, $\mathbf{v}$ is positive, and $\mathbf{v} \circ \mathbf{y} = 0$. By the theory of Lagrange multipliers, the restriction of $f(\mathbf{v}) = \mathbf{v} \circ \mathbf{x}$ to $H$ may attain a local extremum at $\mathbf{v} \in H$ only if the gradient of $f$ at $\mathbf{v}$ is a linear combination of the constraint gradients $\nabla g_1(\mathbf{v})$ and $\nabla g_2(\mathbf{v})$, where $g_1(\mathbf{v}) = \mathbf{v} \circ \mathbf{y}$ and $g_2(\mathbf{v}) = \mathbf{v} \circ \mathbf{v}$. Direct computation yields $\nabla f(\mathbf{v}) = \mathbf{x}$, $\nabla g_1(\mathbf{v}) = \mathbf{y}$, and $\nabla g_2(\mathbf{v}) = 2\mathbf{v}$, where $\mathbf{x}$ is obtained from $\mathbf{v}$ by multiplying the first entry by $-1$ and similarly for the others. It thus follows that $x$ is a linear combination of $\mathbf{y}$ and $\mathbf{v}$ for such a point $\mathbf{v}$, so since $\mathbf{x}$ is not a multiple of $\mathbf{y}$ we can express $\mathbf{v}$ in terms of $\mathbf{x}$ and $\mathbf{y}$.

Plugging $\mathbf{v} = a\mathbf{x} + b\mathbf{y}$ into the constraint equations and solving for $a, b \in \mathbb{R}$ yields two solutions:

$$v = \frac{-1}{\mathbf{x} \circ \mathbf{y}} \mathbf{x} + \mathbf{y}$$

and its opposite. Only one of these is positive, however. Noting that $\mathbf{v} \circ \mathbf{x} = \mathbf{x} \circ \mathbf{y}$ is negative by hypothesis, we conclude that $\mathbf{v}$ is positive and hence is the unique critical point of the restriction of $f$ to $H$. By Lemma 1.2 its signed distance $h$ to $B$ satisfies $e^h = -\mathbf{x} \circ \mathbf{y}$.

The geodesic $\gamma(t)$ through $\mathbf{x}$ in the direction of $\mathbf{x}$ defined in Lemma 1.2 has the form

$$\gamma(t) = e^{-t}v - \frac{\sinh t}{\mathbf{x} \circ \mathbf{v}} \mathbf{x} = \frac{\cosh t}{-\mathbf{x} \circ \mathbf{y}} \mathbf{x} + e^{-t}y.$$ 

For all $t \in \mathbb{R}$, it is clear from this description that $\gamma(t)$ is a positive linear combination of $\mathbf{x}$ and $\mathbf{y}$. Moreover, $\gamma'(0) = -\mathbf{y}$ is not a scalar multiple of $\mathbf{v}$, so the geodesic parametrized by $\gamma$ intersects the Euclidean line through $\mathbf{v}$ and the origin transversely in the plane spanned by $\mathbf{x}$ and $\mathbf{y}$. It therefore has points on either side of this line, so since these points are all positive linear combinations of $\mathbf{x}$ and $\mathbf{y}$, the line separates $\mathbf{x}$ from $\mathbf{y}$ in the plane they span.

The codimension-one subspace of $\mathbb{R}^{n+1}$ containing $H$ intersects the plane spanned by $\mathbf{x}$ and $\mathbf{y}$ in a one-dimensional subspace containing $\mathbf{v}$; that is, in the line through $\mathbf{v}$ and the origin. It
thus follows from the above that this subspace separates \( x \) from \( y \) in \( \mathbb{R}^{n+1} \). Note also that since \( y \) is by construction perpendicular to \( H \), so also is \( \gamma'(0) = -y \).

It remains to show that \( v \) is the global maximizer for the restriction of \( f \) to \( H \), hence that it is the minimizer for the signed distance to \( B \). This follows from the fact that \( v \) is the unique critical point of the restriction of \( f \) to \( H \), together with the fact that \( f(u) \to -\infty \) as \( u \in H \) escapes compact sets. Indeed, for any fixed \( r < 0 \) and \( u \in H \) such that \( u \circ x > r \), we have \( u \circ v = (-1/x \circ y)u \circ x > -r/x \circ y \); hence \( u \) lies in the closed ball of radius \( \cosh^{-1}(r/x \circ y) \) about \( v \).

\[ \square \]

**Lemma 1.6** (cf. [4], pp. 71–75). Let \( x, y \in \mathbb{R}^{n+1} \) be space-like vectors, with polar hyperplanes \( P \) and \( Q \) in \( \mathbb{H}^n \), contained in \( n \)-dimensional subspaces \( V \) and \( W \) of \( \mathbb{R}^{n+1} \), respectively. Exactly one of the following holds:

1. \( P \) and \( Q \) intersect in \( \mathbb{H}^n \), and there exists \( \eta(x, y) \in (0, \pi) \) such that

\[
|x \circ y| = |x||y|\cos \eta(x, y).
\]

Moreover, for any \( z \in P \cap Q \), \( T_zP = V \cap T_z\mathbb{H}^n \), \( T_zQ = W \cap T_z\mathbb{H}^n \), and \( \eta(x, y) \) is the angle in \( T_z\mathbb{H}^n \) between the orthogonal vectors \( x \) and \( y \) to \( T_zP \) and \( T_zQ \), respectively.

2. \( P \) and \( Q \) are disjoint in \( \mathbb{H}^n \), and there exists \( \eta(x, y) \in (0, \infty) \) such that

\[
|x \circ y| = |x||y|\cosh \eta(x, y).
\]

In this case \( \eta(x, y) \) is the distance in \( \mathbb{H}^n \) between \( P \) and \( Q \), and \( x \circ y < 0 \) if and only if \( x \) and \( y \) are opposite each other with respect to the subspace of \( \mathbb{R}^{n+1} \) containing \( P \).

If \( z \) is a positive time-like vector then \( |x \circ z| = |x||z|\sinh d_H(z, P) \), with \( x \circ z < 0 \) if and only if \( x \) and \( z \) are opposite each other with respect to the subspace of \( \mathbb{R}^{n+1} \) containing \( P \).

In the case (2) above, since \( P \) and \( Q \) are convex subsets of \( \mathbb{H}^n \) the distance between them is realized as \( d(z, w) \) for unique \( z \in P \) and \( w \in Q \). The geodesic \( \gamma \) joining \( z \) and \( w \) intersects each of \( P \) and \( Q \) perpendicularly, thus with tangent vector \( x \) at \( z \) and \( y \) at \( w \). It follows that \( \gamma = \text{Span}\{x, y\} \cap \mathbb{H}^n \). Taking \( z = ax + by \) and solving \( z \circ x = 0 \) for \( a \) and \( b \) yields:

\[ z = \pm \frac{(x \circ y/|x|) x - |x||y}}{\sqrt{(x \circ y)^2 - |x|^2|y|^2}} \]

In obtaining the above, we also used that \( z \in \mathbb{H}^n \); hence that \( z \circ z = -1 \). The sign must be chosen, depending on \( x \) and \( y \), so that \( z \) is positive.

2. **Dimension two**

Here we prove trigonometric formulas for a hyperbolic quadrilateral with two ideal vertices and a hyperbolic pentagon with one ideal vertex, each with right angles at all finite vertices.

**Proposition 2.1.** Let \( Q \subset \mathbb{H}^2 \) be a convex quadrilateral with a single compact side of length \( \ell \) and right angles at its endpoints, and let \( B_0 \) and \( B_1 \) be horoballs centered at the two ideal vertices of \( Q \). If \( a_i \) is the signed distance to \( B_i \) from the other endpoint of the half-open edge of \( Q \) containing the ideal point of \( B_i \), \( i = 0, 1 \), and \( d \) is the signed distance from \( B_0 \) to \( B_1 \), then

\[
\sinh(\ell/2) = e^{(d-a_0-a_1)/2}.
\]

If \( \theta_i \) is the length of the horocyclic arc \( S_i \cap Q \), \( i = 0, 1 \), where \( S_i = \partial B_i \), then for each \( i \),

\[
\frac{\theta_0}{e^{a_1}} = \frac{\theta_1}{e^{a_0}} = \frac{\sinh \ell}{2e^d}.
\]
Proof. For a quadrilateral \( Q \subset \mathbb{H}^2 \) with a single compact edge \( \gamma \) and right angles at the endpoints of this edge, let \( x_0 \) and \( x_1 \) be positive light-like vectors determining the horoballs \( B_0 \) and \( B_1 \) centered at the ideal vertices of \( Q \). Using the fact that the geodesic containing \( \gamma \) is a codimension-one hyperplane of \( \mathbb{H}^2 \), let \( y \) be the space-like vector Lorentz-orthogonal to this geodesic with the property that \( x_i \circ y < 0 \) for \( i = 0, 1 \). (Since the ideal vertices of \( Q \) are on the same side of this geodesic, the inner products with \( y \) have the same sign by Lemma 1.5.)

Let \( v_0 \) and \( v_1 \) be the finite vertices of \( Q \), numbered so that \( v_i \) is an endpoint of the half-open edge of \( Q \) with its other endpoint at the center of \( B_i \), for \( i = 0, 1 \). Since \( Q \) is right-angled, \( v_i \) is described in terms of \( x_i \) and \( y \) by the formula (3) for each \( i \). (Note that there is a unique geodesic ray perpendicular to the geodesic containing \( \gamma \) with its ideal endpoint at the center of \( B_i \), since there is no hyperbolic triangle with two right angles.) That is:

\[
\begin{align*}
\text{for } i &= 0, 1, \\
v_0 &= \frac{-1}{x_0 \circ y} x_0 + y \\
v_1 &= \frac{-1}{x_1 \circ y} x_1 + y
\end{align*}
\]

By Lemma 1.5 their signed distances \( a_i \) to the \( B_i \) satisfy \( e^{a_i} = -x_i \circ y \) for \( i = 0, 1 \). If \( \ell \) is the length of \( \gamma \) then from the distance formula we obtain

\[
\cosh \ell = -v_1 \circ v_2 = \frac{-x_0 \circ x_1}{(x_0 \circ y)(x_1 \circ y)} + 1
\]

It follows from Lemma 1.3 that the minimal signed distance \( d \) from \( S_0 \) to \( S_1 \) satisfies \( e^d = -\frac{1}{2}x_0 \circ x_1 \), hence by a half-angle formula \( \sinh(\ell/2) = e^{(d-a_0-a_1)/2} \) as claimed.

Let \( u_0 \) and \( u_0' \) be the points of intersection between the horosphere \( S_0 = \partial B_0 \) and the edges of \( Q \) joining the class of \( x_0 \) to \( v_0 \) and the class of \( x_1 \), respectively. We obtain an explicit description for \( u_0 \) by plugging in \( t = a_0 \) to the parametrized geodesic \( \gamma(t) \) starting at \( v_0 \) given in Lemma 1.5, and for \( u_0' \) by plugging in \( t = d/2 \) to the parametrized geodesic \( \lambda(t) \) from \( x_1 \) given in Lemma 1.3. These yield:

\[
\begin{align*}
u_0 &= \frac{1}{2} \left( 1 + \frac{1}{(x_0 \circ y)^2} \right) x_0 + \frac{-1}{x_0 \circ y} y \\
u_0' &= \frac{1}{2} x_0 + \frac{-1}{x_0 \circ x_1} x_1
\end{align*}
\]

From the horospherical distance formula we thus have

\[
\theta_0 = d_{S_0}(u_0, u_0') = \sqrt{-2(1 + u_0 \circ u_0')} = \sqrt{\frac{1}{(x_0 \circ y)^2} - \frac{2(x_1 \circ y)}{(x_0 \circ x_1)(x_0 \circ y)}}
\]

A similar computation yields an analogous formula for \( \theta_1 \), and we observe that

\[
\theta_0 e^{-a_1} = \theta_1 e^{-a_0} = \sinh \ell / (2e^d)
\]

\[
\frac{1}{(x_0 \circ y)(x_1 \circ y)} \sqrt{\frac{2(x_0 \circ y)(x_1 \circ y) - x_0 \circ x_1}{-x_0 \circ x_1}}
\]

The latter assertion in the statement follows. \( \square \)

**Proposition 2.2.** Let \( P \subset \mathbb{H}^2 \) be a convex pentagon with four right angles and one ideal vertex, and let \( B \) be a horoball centered at the ideal vertex of \( P \). Let \( d \) be the length of the side of \( P \) opposite its ideal vertex, let \( w_0 \) and \( w_1 \) be its endpoints, and for \( i = 0, 1 \) let \( \ell_i \) be the length of the other side containing \( w_i \). If \( v_i \) is the other endpoint of this side and \( a_i \) is its signed distance to \( B \), for \( i = 0, 1 \), then

\[
\cosh \ell_i = \frac{e^{a_i} \cosh d + e^{a_{1-i}}}{e^{a_i} \sinh d} \quad \text{for } i = 0, 1.
\]

Moreover, if \( \theta \) is the length of the horocyclic arc \( S \cap P \), where \( S = \partial B \), then

\[
\frac{\theta}{\sinh d} = \frac{\sinh \ell_0}{e^{a_1}} = \frac{\sinh \ell_1}{e^{a_0}}.
\]
Proof. Let $P$ be a pentagon with four right angles and a single ideal vertex, and let $x$ be a positive light-like vector that determines a horosphere $S$ centered at the ideal vertex of $P$. Labeling the endpoints of the edge of $P$ opposite its ideal vertex as $w_0$ and $w_1$, for $i = 0, 1$ let $\gamma_i$ be the other edge of $P$ containing $w_i$, and let $y_i$ be a unit space-like vector in $\mathbb{R}^3$ orthogonal to the geodesic containing $\gamma_i$. Choose the $y_i$ so that $y_i \cdot x < 0$ for each $i$. Equivalently, by Lemma 1.5, $y_i$ is on the opposite side of $x$ from the plane $u \cdot x = 0$ in $\mathbb{R}^3$. Since $\gamma_i$ and the ideal point of $P$ are on the same side of the geodesic containing $\gamma_{1-i}$ for each $i$, $y_0 \cdot y_1 < 0$ by [4].

Let us call $v_i$ the endpoint of $\gamma_i$ not equal to $w_i$, for $i = 0, 1$. An explicit formula for $v_i$ is given by (3), with $y$ there replaced by $y_i$. As in the proofs of Theorem 3.2.7 and 3.2.8 of [4] we have the following explicit formula for $w_i$:

$$w_i = \frac{-(y_0 \cdot y_1)y_i + y_{1-i}}{\pm \sqrt{(y_0 \cdot y_1)^2 - 1}},$$

where “+” or “−” is chosen so that $w_i$ is a positive vector. For, say, $i = 0$ we thus have

$$w_0 \cdot v_0 = \frac{y_0 \cdot y_1 - (x \cdot y_1)/(x \cdot y_0)}{\pm \sqrt{(y_0 \cdot y_1)^2 - 1}} = \frac{-(x \cdot y_0)(y_0 \cdot y_1) + x \cdot y_1}{-(x \cdot y_0)\sqrt{(y_0 \cdot y_1)^2 - 1}}$$

In passing from the first to the second equality we have fixed the sign choice “+” for the radical. This is the right choice since $y_0 \cdot y_1$ and the $x \cdot y_i$ are all negative, and $w_0 \cdot v_0$ is as well.

If $\ell_i$ is the length of $\gamma_i$ and $a_i$ is the distance from $v_i$ to $S$, for $i = 0, 1$, and $d = d_H(w_0, w_1)$ is the length of the side opposite the ideal vertex, then the above equation becomes

$$\cosh \ell_0 = \frac{e^{a_0} \cosh d + e^{a_1}}{e^{a_0} \sinh \ell}$$

This is because $\cosh \ell = -w_0 \cdot v_0$ by definition, $d_H(v_i, S) = -x \cdot y_i$ by Lemma 1.2, and as can be explicitly checked, $\cosh d = -w_0 \cdot w_1 = -y_0 \cdot y_1$. The derivation of the formula for $\cosh \ell_1$ is analogous, and we have proved the hyperbolic law of cosines.

For the law of sines we first note that the point of intersection $u_i$ between $S$ and the geodesic from $v_i$ in the direction of $x$ is given by the formula (1), with $v$ there replaced by $v_i$, for $i = 0, 1$. From direct calculation and/or Lemma 1.5 we have $v_i \cdot x = y_i \cdot x$, whence for each $i$ we have

$$u_i = \frac{1}{2} \left(1 + \frac{1}{(x \cdot y_i)^2}\right)x + \frac{-1}{x \cdot y_i}y_0$$

From this we obtain the following formula for the length $\theta$ of the horocyclic arc $S \cap P$:

$$\theta = \sqrt{-2(1 + u_0 \cdot u_1)} = \sqrt{(x \cdot y_0)^2 + (x \cdot y_1)^2 - 2(y_0 \cdot y_1)(x \cdot y_0)(x \cdot y_1)}$$

Direct computation now establishes this case of the hyperbolic law of sines. \hfill \square

3. Dimension three: Transversals of truncated tetrahedra

Turning our attention to $\mathbb{H}^3$, we recall that three disjoint totally geodesic planes $P_1$, $P_2$, and $P_3$ in $\mathbb{H}^3$ determine a unique plane $P$ with the property that for each $i$, $P_i$ intersects $P$ at right angles (see eg. [2, Lemma 2.3]). Suppose in addition that for each $i$, a single half-space bounded by $\mathbb{H}^3 - P_i$ contains the other two planes. Then there is a right-angled hexagon $C$ in $P$ with alternating sides consisting of the three mutual perpendiculars $\lambda_{ij}$ joining $P_i$ to $P_j$, $1 \leq i < j \leq 3$, such that each other side of $C$ lies in one of the sides $P_i$, $1 \leq i \leq 3$.

Now consider four disjoint totally geodesic planes $P_1$, $P_2$, $P_3$, $P_4$ in $\mathbb{H}^3$, with the property that for each $i$, a single half-space $H_i$ bounded by $P_i$ contains the other three planes. For any $i$, let $\tilde{P}_i$ be the mutual perpendicular plane, as above, to the other three planes. If for some $i \neq i'$, $\tilde{P}_i$ does not coincide with $\tilde{P}_{i'}$, then the natural geometric object associated to the four planes is
a truncated tetrahedron. This is the intersection of the four half-spaces \( H_i \) with those bounded by the planes \( P_i \) and containing the fourth plane. It is homeomorphic to the complement in a tetrahedron of the union of small regular neighborhoods of the vertices; see Figure 1.

The truncated tetrahedron determined by four disjoint planes as above is also specified by the quadruple of vectors \((x_1, x_2, x_3, x_4)\) such that \(x_i\) spans the Lorentz orthogonal complement to \(P_i\) for each \(i, 1 \leq i \leq 4\). Conversely, any truncated tetrahedron determines such a quadruple, after making the following additional requirements: that \(x_i \circ x_i = 1\) for each \(i\), \(x_i \circ x_j < 0\) for each \(j \neq i\), and that

\[
z_1 = \frac{(x_1 \circ x_2)x_1 - x_2}{\sqrt{(x_1 \circ x_2)^2 - 1}}
\]

is positive. (Recall from (4) that \(z_1 = P_1 \cap \lambda_{12}\), where \(\lambda_{12}\) is the mutual perpendicular in \(\mathbb{H}^3\) to \(P_1\) and \(P_2\).) For \(1 \leq i < j \leq 4\), let \(L_{ij} = -(x_i \circ x_j)\). By Lemma 1.6 and the hypotheses on the \(x_i\), for each such \(i, j\), \(L_{ij} = \cosh d_{H}(P_i, P_j)\). Then \(z_1 = -[L_{12}x_1 + x_2]/\sqrt{L_{12}^2 - 1}\), and taking \(z_3 = -[L_{34}x_3 + x_4]/\sqrt{L_{34}^2 - 1}\), we find:

\[
z_1 \circ z_3 = -\frac{L_{12}L_{13}L_{34} + L_{12}L_{14} + L_{23}L_{34} + L_{24}}{\sqrt{(L_{12}^2 - 1)(L_{34}^2 - 1)}} < 0
\]

It follows that \(z_3\) is positive and thus is \(P_3 \cap \lambda_{34}\), where \(\lambda_{34}\) is the mutual perpendicular in \(\mathbb{H}^3\) to \(P_3\) and \(P_4\). We will be interested in how the distance between the “opposite” perpendiculars \(\lambda_{12}\) and \(\lambda_{34}\) varies with the \(L_{ij}\). Recall a parametrization for \(\lambda_{12}\):

\[
\lambda_{12}(s) = \cosh s \ z_1 + \sinh s \ x_1, \ s \in \mathbb{R}.
\]

This parametrization gives \(\lambda_{12}(0) = z_1\), and since it is by arclength, \(\lambda_{12}(d_{H}(P_1, P_2)) \in P_2\). (Indeed, it can be checked that \(x_2 \circ \lambda_{12}(d_{H}(P_1, P_2)) = 0\).) Similarly, we parametrize \(\lambda_{34}\) by \(\lambda_{34}(t) = \cosh t \ z_3 + \sinh t \ x_3\) for \(0 \leq t \leq d_{H}(P_3, P_4)\). The distance between \(\lambda_{12}(s)\) and \(\lambda_{34}(t)\) satisfies \(\cosh d_{H}(\lambda_{12}(s), \lambda_{34}(t)) = D(s, t)\), where

\[
D(s, t) = -(\cosh s \ z_1 + \sinh s \ x_1) \circ (\cosh t \ z_3 + \sinh t \ x_3)
\]

\[
= \cosh s \cosh t \ L_{12}L_{13}L_{34} + L_{12}L_{14} + L_{23}L_{34} + L_{24} \sqrt{(L_{12}^2 - 1)(L_{34}^2 - 1)}
\]

\[
- \cosh s \sinh t \frac{L_{12}L_{13} + L_{23}}{\sqrt{L_{12}^2 - 1}} - \sinh s \cosh t \frac{L_{13}L_{34} + L_{14}}{\sqrt{L_{34}^2 - 1}} + \sinh s \sinh t \ L_{13}.
\]

Given a truncated tetrahedron determined by disjoint planes \(P_i\), \(1 \leq i \leq 4\), and parametrizations as above for the geodesics \(\lambda_{12}\) and \(\lambda_{34}\), there exist unique \(s_0\) and \(t_0\) minimizing \(d_{H}(\lambda_{12}(s), \lambda_{34}(t))\). This is because \(\lambda_{12}\) and \(\lambda_{34}\) are convex subsets of \(\mathbb{H}^3\). The geodesic \(\gamma\) joining \(\lambda_{12}(s_0)\) to \(\lambda_{34}(t_0)\) meets each of \(\lambda_{12}\) and \(\lambda_{34}\) perpendicularly. It follows that \(0 < s_0 < d_{H}(P_1, P_2)\) and \(0 < t_0 <
$d_H(P_3, P_4)$, since for instance any geodesic meeting $\gamma_{12}$ perpendicularly at $\gamma_{12}(s)$ for $s < 0$ is on the other side of $P_1$ from $\gamma_{34}$.

**Lemma 3.1.** Given a truncated tetrahedron $T$ determined by planes $P_i$, $1 \leq i \leq 4$, let $L_{ij} = \cosh d_H(P_i, P_j)$ for $1 \leq i < j \leq 4$.

(1) If for some $L > 1$, $L_{ij} = L$ for all $i \neq j$, then:

$$\cosh d_H(\lambda_{12}, \lambda_{34}) = \frac{2L}{L - 1}$$

(2) If $L_{12} = L_{13} = L_{14} = L_{23} = L_1$ and $L_{34} = L_2$, then:

$$\cosh d_H(\lambda_{12}, \lambda_{34}) = \frac{2L_1}{\sqrt{(L_1 - 1)(L_2 - 1)}}$$

(3) If $L_{13} = L_{14} = L_{23} = L_{24} = L_1$ and $L_{12} = L_{34} = L_2$, then:

$$\cosh d_H(\lambda_{12}, \lambda_{34}) = \frac{2L_1}{L_2 - 1}$$

**Proof.** Let $\gamma$ be the geodesic that intersects each of $\lambda_{12}$ and $\lambda_{34}$ perpendicularly. The key observation here is that in each case above, the rotation by $\pi$ about $\gamma$ restricts to a symmetry of $T$. This is because of the combinatorial symmetry in the lengths $L_{ij}$. In particular, the values $s_0$ and $t_0$ that minimize $D$ are $\frac{1}{2} d_H(P_1, P_2)$ and $\frac{1}{2} d_H(P_3, P_4)$, respectively.

Given this observation, establishing the result is a matter of computation. We record cases (2) and (3). In case (2), since $\cosh 2s_0 = L_{12} = L_1$, the “half-angle formula” for the hyperbolic cosine yields $\cosh s_0 = \sqrt{\frac{L_1 + 1}{2}}$ and $\sinh s_0 = \sqrt{\frac{L_1 - 1}{2}}$. Since $\cosh 2t_0 = L_{34} = L_2$, we analogously have $\cosh t_0 = \sqrt{\frac{L_2 + 1}{2}}$ and $\sinh t_0 = \sqrt{\frac{L_2 - 1}{2}}$. Substituting into the formula for $D$, we obtain:

$$D(s_0, t_0) = \frac{L_1^2 L_2 + L_1^2 + L_1 L_2 + L_1}{2\sqrt{(L_1 - 1)(L_2 - 1)}} - \frac{\sqrt{L_2 - 1} L_1^2 + L_1}{L_1 - 1} - \frac{\sqrt{L_2 - 1} L_1 L_2 + L_1}{L_2 - 1} + \sqrt{(L_1 - 1)(L_2 - 1)} \frac{L_1}{2}$$

$$= \frac{L_1}{2\sqrt{(L_1 - 1)(L_2 - 1)}} \left( L_1 L_2 + L_1 + L_2 + 1 - (L_2 - 1)(L_1 + 1) \right)$$

$$= \frac{2L_1}{\sqrt{(L_1 - 1)(L_2 - 1)}}$$

In case (3), since $\cosh 2s_0 = \cosh 2t_0 = L_2$, we have $\cosh s_0 = \cosh t_0 = \sqrt{\frac{L_2 + 1}{2}}$ and $\sinh s_0 = \sinh t_0 = \sqrt{\frac{L_2 - 1}{2}}$. Substituting into the formula for $D$, we obtain:

$$D(s_0, t_0) = \frac{L_1 L_2^2 + 2L_1 L_2 + L_1}{2(L_2 - 1)} - \frac{L_1 L_2 + L_1}{2} - \frac{L_1 L_2 + L_1}{2} + \frac{L_1 (L_2 - 1)}{2}$$

$$= \frac{L_1}{2(L_2 - 1)} \left[ L_2^2 + 2L_2 + 1 - 2(L_2^2 - 1) + L_2^2 - 2L_2 + 1 \right] = \frac{2L_1}{L_2 - 1}$$

Case (1) is similar. □

Fixing $s_0$, $t_0$, and all but one of the $L_{ij}$ for the moment, we will regard $D$ as a function of $L_{24}$, and note that

$$D'(L_{24}) = \frac{\cosh s_0 \cosh t_0}{\sqrt{(L_{12}^2 - 1)(L_{34}^2 - 1)}} > 0.$$
It follows that for for two tetrahedra with equal $L_{ij}$ for $ij \neq 24$, the one with larger $L_{24}$ will determine a larger minimum value for $D$, hence a larger value for $d_H(\lambda_{12}, \lambda_{34})$. We record this below.

**Lemma 3.2.** Let $T = T(\ell, \ell', a, b, c, d)$ and $T' = T(\ell, \ell', a', b, c, d)$ be truncated tetrahedra, where $a > a'$. Then $T$ has a longer transversal length than $T'$.

We now change the set-up slightly by replacing the plane $P_4$ with a horoball $B$ disjoint from $P_1, P_2,$ and $P_3$, and such that for each $i \in \{1, 2, 3\}$, $P_i$ bounds a half-space $H_i$ containing $B$ and the other two hyperplanes.

If the ideal point of $B$ does not lie in the mutual perpendicular $\hat{P}$ to $P_1, P_2,$ and $P_3$, then we define the partially truncated tetrahedron determined by $B$ and the $P_i$ to be the intersection of the $H_i, i = 1, 2, 3,$ with the half-space $H$ bounded by $\hat{P}$ that contains the ideal point of $B$, and three half-spaces $\hat{H}_i, i = 1, 2, 3.$ For each such $i$, $\hat{H}_i$ is bounded by the mutual perpendicular $\hat{P}_i$ to the other two planes $P_j, P_k$ that contains the ideal point of $B$ and hence meets $B$ perpendicularly. $\hat{H}_i$ is the half-space bounded by $P_i$ that contains the shortest geodesic arc from $B$ to $P_i$.

**Definition 3.3.** Taking $a, b$ and $c$ to be the distances from $P_1$ to $P_2, P_2$ to $P_3$ and $P_3$ to $P_1$, respectively, and $h_i$ to be the distance from $P_i$ to $B$, for each $i \in \{1, 2, 3\},$ denote the partially truncated tetrahedron constructed as above by $T(h_1, h_2, h_3, a, b, c)$.

**Proposition 3.4.** For $T \doteq T(h_1, h_2, h_3, a, b, c)$ as in Definition 3.3, if $h_i = h$ for each $i$, and $a = b = c = \ell_1$, for fixed $h$ and $\ell_1 > 0$, then the distance $D$ between the edge of $T$ joining $P_1$ to $B$ and the edge joining $P_2$ to $P_3$ satisfies

$$
cosh D = 2 \sqrt{1 + \frac{\cosh \ell_1 \sqrt{2}}{\cosh \ell_1 - 1}}.
$$

**Proof.** Let $x \in \mathbb{R}^{1,3}$ be the positive light-like vector that determines the horoball $B$, and for $i = 1, 2, 3$ let $y_i$ be a unit space-like vector in $\mathbb{R}^3$ normal to $P_i$ and such that $x \circ y_i < 0$ for each $i$. Then also, $y_i \circ y_j < 0$ for $j \neq i$ by hypothesis.

By the proof of Lemma 1.5, the geodesic ray $\gamma(t)$ from the closest point $v_1$ on $P_1$ in the direction of $x$ satisfies

$$
\gamma(t) = e^{-t} v_1 - \frac{\sinh t}{x \circ v_1} x = \frac{\cosh t}{-x \circ y_1} x + e^{-t} y_1.
$$

We wish to minimize the distance from $\gamma(t)$ to the geodesic from $P_2$ to $P_3$. Taking $L_1 = \cosh \ell_1$ we recall that this is parametrized as

$$
\lambda_{23}(s) = \cosh s z_2 + \sinh s y_2, \text{ for } z_2 = \frac{(y_2 \circ y_3)y_2 - y_3}{\sqrt{(y_2 \circ y_3)^2 - 1}} = \frac{L_1 y_2 - y_3}{\sqrt{L_1^2 - 1}}.
$$

Due to the symmetry of the situation, the closest point of $\lambda_{23}$ to $\gamma(t)$ is its midpoint $s_0$, which satisfies $\cosh s_0 = \sqrt{L_1 + 1}$ and $\sinh s_0 = \sqrt{L_1 - 1}/2$. Plugging this in gives $\gamma(s_0) = (-y_2 - y_3)/\sqrt{2(L_1 - 1)}$. We thus are looking to minimize

$$
\gamma(t) \circ \lambda_{23}(s_0) = \frac{x \circ y_2 + x \circ y_3}{x \circ y_1} \cosh t - \frac{y_1 \circ y_2 + y_1 \circ y_3}{\sqrt{2(L_1 - 1)}} e^{-t} = 2 \cosh t + \frac{2L_1}{\sqrt{2(L_1 - 1)}} e^{-t} = e^t + \left(1 + \frac{L_1 \sqrt{2}}{\sqrt{L_1 - 1}}\right) e^{-t}
$$

**Proof.**}
Setting a derivative equal to 0 yields
\[
e^{2t} = 1 + \frac{L_1 \sqrt{2}}{\sqrt{L_1 - 1}} \Rightarrow e^t = \sqrt{1 + \frac{L_1 \sqrt{2}}{\sqrt{L_1 - 1}}}
\]
Plugging this into \( \gamma(t) \circ \lambda_{23}(s_0) \) yields the formula given above. \( \square \)

**Lemma 3.5.** For \( T = T(h_1, h_2, h_3, a, b, c) \) as in Definition 3.3, if \( h_i \geq h \) for each \( i \), and \( a, b, \) and \( c \) are all at least \( \ell_1 \), for fixed \( h \) and \( \ell_1 > 0 \), then the distance \( D \) between the edge of \( T \) joining \( P_1 \) to \( B \) and the edge joining \( P_2 \) to \( P_3 \) is at least \( D \) from Proposition 3.4.

We now replace another plane by a horoball. That is, consider a collection \( P_1, P_2, B_1, B_2 \) of mutually disjoint planes (the \( P_i \)) and horoballs (the \( B_i \)) such that for each \( i \in \{1, 2\}, B_1, B_2, \) and \( P_{3-i} \) are contained in a single complementary component of \( P_i \). As in the previous case, there is a partially truncated tetrahedron determined by the \( P_i, i = 1, 2 \), and the four planes orthogonal to triples of the four vertex objects.

**Definition 3.6.** Taking \( \ell \) be the distance from \( P_1 \) to \( P_2 \), \( d \) the distance from \( B_1 \) to \( B_2 \), and \( h_{ij} \) the distance from \( P_i \) to \( P_j \), for \( i, j \in \{1, 2\} \), denote the partially truncated tetrahedron constructed above by \( T(d, h_{11}, h_{12}, h_{21}, h_{22}, \ell) \).

**Proposition 3.7.** For \( T = T(d, h_{11}, h_{12}, h_{21}, h_{22}, \ell) \) as in Definition 3.6, if \( h_{ij} = h \) for each \( i, j \in \{1, 2\} \), then the distance \( D \) from the edge of \( T \) joining \( P_1 \) to \( B_1 \) to the edge joining \( P_2 \) to \( B_2 \) satisfies
\[
cosh D = 1 + \frac{e^d}{e^{2h}} + \sqrt{\frac{e^d}{e^{2h}}} \sqrt{2 + 2 \cosh \ell} + \frac{e^d}{e^{2h}}.
\]

**Proof.** For \( i = 1, 2 \), let \( x \in \mathbb{R}^{1,3} \) be the positive light-like vector that determines the horoball \( B_i \), and let \( y_i \) be a unit space-like vector in \( \mathbb{R}^3 \) normal to \( P_i \) and such that \( x_i \circ y_j < 0 \) for each \( i \). For each \( i \), the geodesic ray \( \gamma_i \) from the closest point \( v_i \) of \( P_i \) in the direction of \( x_i \) is given by
\[
\gamma_i(t) = \frac{\cosh t}{-x_i \circ y_i} x_i + e^{-t} y_i.
\]
Let \( \lambda \) be the mutual perpendicular to the geodesic joining the ideal point of \( B_1 \) to that of \( B_2 \) and the geodesic containing the shortest arc between \( P_1 \) and \( P_2 \). The \( \pi \)-rotation around \( \lambda \) exchanges \( B_i \) with \( B_{3-i} \) and \( P_i \) with \( P_{3-i} \), for \( i = 1, 2 \). Thus it takes \( \gamma_i(s) \) to \( \gamma_{3-i}(s) \) for \( i = 1, 2 \). It follows that the shortest distance between \( \gamma_1(s) \) and \( \gamma_2(t) \) is realized at some \( s = t \). To identify this \( t \), we set \( \frac{d}{dt} [-\gamma_1(t) \circ \gamma_2(t)] \) equal to 0. After simplification, this yields:
\[
e^{2t} = \sqrt{\frac{2 + 2 \cosh \ell_1 + e^d/e^{2h}}{e^{d}/e^{2h}}}
\]
Plugging this back into \( -\gamma_1(t) \circ \gamma_2(t) \) yields the result. \( \square \)

**References**


