DIFFERENTIAL GEOMETRY 2, HOMEWORK 3 ADDENDUM

(1) Suppose $F\colon M \to N$ is a smooth map of smooth manifolds, and g is a Riemannian metric on N. For $p \in M$, let $\phi = (x^1, \ldots, x^m)\colon U \to \mathbb{R}^m$ be a chart on M with $p \in U$, and let $\psi = (y^1, \ldots, y^n)\colon V \to \mathbb{R}^n$ be a chart on N with $F(p) \in V$. If

$$g_{F(p)} = \sum_{i,j=1}^{n} g_{ij}(F(p))\, dy^i|_{F(p)} \otimes dy^j|_{F(p)}$$

then verify for F^*g as defined in class that

$$(F^*g)_p = \sum_{i,j=1}^{m} \left[\sum_{k,l=1}^{n} g_{kl}(F(p)) \frac{\partial f^k}{\partial x^i}|_p \frac{\partial f^l}{\partial x^j}|_p \right] dx^i|_p \otimes dx^j|_p,$$

where $f^k = y^k \circ F$ and $f^l = y^l \circ F$ for all k and l with $1 \leq k, l \leq n$.

(2) (John M. Lee, *Riemannian Manifolds*, Problem 4-3)

(a) Show that the Lie bracket (aka Lie derivative) $(\mathbf{X}, \mathbf{Y}) \mapsto \mathcal{L}_{\mathbf{X}}\mathbf{Y} \doteq [\mathbf{X}, \mathbf{Y}]$ on a smooth manifold M is not a connection on M.

(b) Show that there are vector fields $\mathbf{V}$ and $\mathbf{W}$ on $\mathbb{R}^2$ with $\mathbf{V} = \mathbf{W} \equiv \frac{\partial}{\partial x^1}$ along the x-axis but with $\mathcal{L}_{\mathbf{V}}\left(\frac{\partial}{\partial x^2}\right) \neq \mathcal{L}_{\mathbf{W}}\left(\frac{\partial}{\partial x^2}\right)$ along the x-axis. (This shows that Lie differentiation does not give a well-defined way to take derivatives of vector fields along curves.)

(3) (*Riemannian Manifolds*, Problem 5-4) Recall that a vector field $\mathbf{V}$ on a manifold M is *parallel* if $\nabla_{\mathbf{X}}\mathbf{V}|_p = 0$ for every $p \in M$ and $\mathbf{X} \in T_pM$.

(a) Let $p \in \mathbb{R}^n$ and $\mathbf{V}_p \in T_p\mathbb{R}^n$. Show that $\mathbf{V}_p$ has a unique extension to a parallel vector field $\mathbf{V}$ on $\mathbb{R}^n$.

(b) Let U be the open subset of the sphere S^2 on which (German-style) spherical coordinates (θ, ϕ) are defined, and let $\mathbf{V} = \frac{\partial}{\partial \phi}$ in these coordinates. Compute $\nabla_{\frac{\partial}{\partial \theta}}\mathbf{V}$ and $\nabla_{\frac{\partial}{\partial \phi}}\mathbf{V}$ and conclude that $\mathbf{V}$ is parallel along the equator and each meridian $\theta = \theta_0$.

(c) Let $p = (0, 0)$ in spherical coordinates, and show that $\mathbf{V}_p$ has no parallel extension to any neighborhood of p.

(d) Use (a) and (c) to show that no neighborhood of p is isometric to an open subset of $\mathbb{R}^2$.