DIFFERENTIAL GEOMETRY 2, HOMEWORK 4 ADDENDUM

Problem 1. Prove parts (1) and (assuming (2)) (3) of the proposition below.

Proposition (Prop. 5.6, *Riemannian Manifolds*)**.** *For an isometry $\phi\colon (M, g) \to (\widetilde{M}, \tilde{g})$ of Riemannian manifolds:*

(1) *ϕ takes the Riemannian connection ∇ of g to the Riemannian connection $\tilde{\nabla}$ of $\tilde{g}$, in the sense that for all vector fields $\mathbf{X}$ and $\mathbf{Y}$ on M,*

$$\phi_*(\nabla_{\mathbf{X}}\mathbf{Y}) = \tilde{\nabla}_{\phi_*\mathbf{X}}(\phi_*\mathbf{Y}),$$

Here $\phi_*\mathbf{X}$ is the push-forward *of $\mathbf{X}$, the vector field on $\widetilde{M}$ defined at $p \in \widetilde{M}$ by $(\phi_*\mathbf{X})_p = D\phi|_{\phi^{-1}(p)}(\mathbf{X}_{\phi^{-1}(p)})$.*

(2) *If c is a curve in M and $\mathbf{V}$ is a vector field along c then taking $\tilde{c} = \phi \circ c$,*

$$\phi_*\nabla_{\dot{c}}\mathbf{V} = \tilde{\nabla}_{\dot{\tilde{c}}}(\phi_*\mathbf{V}),$$

where $(\phi_*\mathbf{V})_t = D\phi|_{c(t)}(\mathbf{V}_t)$ and $\nabla_{\dot{c}}$ is the covariant derivative along c.

(3) *ϕ takes geodesics to geodesics.*

Hint: To prove part (1), show that the *pullback* $\phi^*\widetilde{\nabla}$ of $\widetilde{\nabla}$ to M, defined by

$$(\phi^*\widetilde{\nabla})_{\mathbf{X}}\mathbf{Y} = \phi_*^{-1}\left((\widetilde{\nabla}_{\phi_*\mathbf{X}}(\phi_*\mathbf{Y})\right)$$

is a torsion-free connection on M which is compatible with g. You may use the following properties of the push-forward:

Lemma. *Let $\mathbf{X}$ be a smooth vector field on a Riemannian manifold (M, g) and $\phi\colon (M, g) \to (\widetilde{M}, \tilde{g})$ an isometry of Riemannian manifolds. If $\mathbf{X} = \sum_i \xi^i \frac{\partial}{\partial x^i}$ in local coordinates $(x^1, \ldots, x^n)$ around some $p \in M$ then*

$$\phi_*\mathbf{X} = \sum_{i,j} \left(\xi^j \circ \phi^{-1}\right) \left(\frac{\partial \phi^i}{\partial x^j} \circ \phi^{-1}\right) \frac{\partial}{\partial y^i}$$

in local coordinates $(y^1, \ldots, y^n)$ about $\phi(p)$. Moreover, for any smooth functions f and g and vector field $\mathbf{Y}$ on M, the push-forward satisfies:

Functoriality: *For an isometry ψ defined on $\widetilde{M}$, $(\psi \circ \phi)_*\mathbf{X} = \psi_*(\phi_*\mathbf{X})$. In particular, $\phi_*^{-1}\phi_*\mathbf{X} = \mathbf{X}$.*

Linearity over $C^\infty M$: $\phi_*(f\mathbf{X} + g\mathbf{Y}) = (f \circ \phi^{-1})\phi_*\mathbf{X} + (g \circ \phi^{-1})\phi_*\mathbf{Y}$

Compatibility with the Lie bracket: $\phi_*[\mathbf{X}, \mathbf{Y}] = [\phi_*\mathbf{X}, \phi_*\mathbf{Y}]$

Compatibility with the metric: *For any $\tilde{p} \in \widetilde{M}$, if $p = \phi^{-1}(\tilde{p})$ then*

$$\tilde{g}_{\tilde{p}}(\phi_*\mathbf{X}_{\tilde{p}}, \phi_*\mathbf{Y}_{\tilde{p}}) = g_p(\mathbf{X}_p, \mathbf{Y}_p)$$

Proof. The local coordinate description of $\phi_*\mathbf{X}$ is a direct consequence of the definition and the following fact proven in class: for local coordinates $(x^1,\dots,x^n)$ and $(y^1,\dots,y^n)$ around p and $\phi(p)$, respectively, and a tangent vector $\sum_i \xi^i \frac{\partial}{\partial x^i}|_p$ at p,

$$D\phi|_p\left(\sum_i \xi^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum_{i,j} \xi^j \frac{\partial \phi^i}{\partial x^j}(p) \left.\frac{\partial}{\partial y^i}\right|_{\phi(p)},$$

where $\phi^i = y^i \circ \phi$.

Functoriality follows from the definition of the push-forward and the chain rule. Linearity over $C^\infty M$ follows from the definition and the fact that for each $p \in M$, $D\phi|_p$ is a linear transformation. Compatibility with the metric also follows by checking the definition and using the fact that ϕ is an isometry. Compatibility with the Lie bracket is more complicated. You can check it by a massive computation in local coordinates (again crucially using the chain rule), or via the direct argument below.

Recall that if $\mathbf{X}_p$ is regarded as a derivation on germs at p of smooth functions on M, then $D\phi|_p(\mathbf{X}_p)$ is the derivation on germs at $\phi(p)$ of smooth functions on $\widetilde{M}$ defined by $D\phi|_p(\mathbf{X}_p)(f) = \mathbf{X}_p(f\circ\phi)$. From the derivation perspective the Lie bracket $[\mathbf{X},\mathbf{Y}]$ is defined by:

$$[\mathbf{X},\mathbf{Y}]_p(f) = \mathbf{X}_p\left(q \mapsto \mathbf{Y}_q(f)\right) - \mathbf{Y}_p\left(q \mapsto \mathbf{X}_q(f)\right)$$

for the germ at p of a smooth function f on M, where $q \mapsto \mathbf{Y}_q(f)$ refers to the germ at p of the function on M taking q to $\mathbf{Y}_q(f)$. Then for $\tilde{p} \in \widetilde{M}$ and the germ at $\tilde{p}$ of a smooth function f on $\widetilde{M}$, by definition we have

$$\begin{aligned}\phi_*[\mathbf{X},\mathbf{Y}]_{\tilde{p}}(f) &= D\phi|_p\left([\mathbf{X},\mathbf{Y}]_p\right)(f) = [\mathbf{X},\mathbf{Y}]_p(f\circ\phi)\\ &= \mathbf{X}_p(q\mapsto \mathbf{Y}_q(f\circ\phi)) - \mathbf{Y}_p(q\mapsto \mathbf{Y}_q(f\circ\phi)),\end{aligned}$$

where $p = \phi^{-1}(\tilde{p})$. With the same notation we have on the other hand:

$$\begin{aligned}[\phi_*\mathbf{X},\phi_*\mathbf{Y}]_{\tilde{p}}(f) &= (\phi_*\mathbf{X})_{\tilde{p}}(\tilde{q}\mapsto(\phi_*\mathbf{Y})_{\tilde{q}}(f)) - (\phi_*\mathbf{Y})_{\tilde{p}}(\tilde{q}\mapsto(\phi_*\mathbf{X})_{\tilde{q}}(f))\\ &= \mathbf{X}_p\left((\tilde{q}\mapsto \mathbf{Y}_{\phi^{-1}(\tilde{q})}(f\circ\phi))\circ\phi\right)\\ &\qquad - \mathbf{Y}_p\left((\tilde{q}\mapsto \mathbf{Y}_{\phi^{-1}(\tilde{q})}(f\circ\phi))\circ\phi\right)\end{aligned}$$

The point I'm trying to emphasize with this notation is that the inputs for the function $\tilde{q}\mapsto \mathbf{Y}_{\phi^{-1}(\tilde{q})}(f\circ\phi)$ are points of $\widetilde{M}$. Upon composing this function with ϕ one simply obtains $q\mapsto \mathbf{Y}_q(f\circ\phi)$, since $\phi^{-1}(\phi(q)) = q$. Compatibility of the push-forward with the Lie bracket follows. □