DIFFERENTIAL GEOMETRY 2, HOMEWORK 5 ADDENDUM

(1) For a Riemannian manifold (M, g) and a smooth one-form ω on M, prove that there is a unique smooth vector field $\omega^\#$ on M with the property that

$$\omega_p(\mathbf{Y}_p) = g_p(\mathbf{Y}_p, \omega_p^\#)$$

for all $p \in M$ and $\mathbf{Y}_p \in T_pM$. Show moreover that in local coordinates, if $\omega = \sum_j \alpha_j \, dx^j$ then $\omega^\# = \sum_{i,j} g^{ij} \alpha_j \frac{\partial}{\partial x^i}$.

(2) *Spherical coordinates* on $\mathbb{R}^3$ are the inverse of the map

$$(\rho, \theta, \phi) \mapsto (\rho \cos\phi \cos\theta, \rho \cos\phi \sin\theta, \rho \sin\phi),$$

defined on the image of $(0, \infty) \times (0, 2\pi) \times (-\pi/2, \pi/2)$. *Cylindrical coordinates* are the inverse of

$$(r, \theta, z) \mapsto (r \cos\theta, r \sin\theta, z),$$

again only defined on a subset of $\mathbb{R}^3$: the image of $(0, \infty) \times (0, 2\pi) \times \mathbb{R}$. In each of cylindrical and spherical coordinates, calculate:

(a) the coefficients of the standard inner product g;
(b) the Christoffel symbols of g; and
(c) the coefficients of the Riemann curvature tensor.