DIFFERENTIAL GEOMETRY 1, HOMEWORK 4

(1) Chapter 2, Exercise 10.

(2) Given a finite open cover $\mathcal{V} = \{V_1, \ldots, V_n\}$ of a topological space X and a continuous map $c\colon [a,b] \to X$, prove there is a *partition of* $[a,b]$ *adapted to* $\mathcal{V}$: a collection $a = t_0 < t_1 < \ldots < t_m = b$ such that for each $i > 0$, $c([t_{i-1}, t_i]) \subset V_j$ for some $j = j(i)$.

(3) Show that every regular closed curve $c\colon [a,b] \to \mathbb{R}^2$ has a tangent line ℓ with the property that $c([a,b])$ is entirely contained on one side of ℓ.

(4) Given a continuous function $e\colon A \to \mathbb{R}^2 - \{\mathbf{0}\}$ on a set $A \subset \mathbb{R}^2$ that is star-like with respect to some $x_0 \in A$, show for any fixed $\mathbf{x} \in A$ that there exists $\delta > 0$ such that for all $\mathbf{y} \in A$ with $\|\mathbf{x} - \mathbf{y}\| < \delta$, $c_{\mathbf{x}}(t)$ and $c_{\mathbf{y}}(t)$ are not antipodal for all $t \in [0,1]$. Here $c_{\mathbf{x}}(t) = e(t\mathbf{x} + (1-t)\mathbf{x}_0)$, and analogously for $c_{\mathbf{y}}$.

(5) For a simply closed, regular curve $c\colon [a,b] \to \mathbb{R}^2$, taking $A = \{(s,t) \mid a \le s \le t \le b\}$ show that the function $e\colon A \to \mathbb{R}^2 - \{\mathbf{0}\}$ defined below is continuous:

$$e(s,t) = \begin{cases} \frac{c(t)-c(s)}{\|c(t)-c(s)\|} & s \neq t, (s,t) \neq (a,b) \\ \frac{\dot{c}(t)}{\|\dot{c}(t)\|} & s = t \\ -\frac{\dot{c}(a)}{\|\dot{c}(a)\|} & (s,t) = (a,b) \end{cases}$$

(6) Describe a non-simple regular closed plane curve with everywhere-positive curvature.