DIFFERENTIAL GEOMETRY I
POSSIBLE FINAL PROJECT TOPICS

The list below may grow as things occur to me, and I will insert good references as I discover them (published papers may be found on mathscinet: `www.ams.org/mathscinet`). Of course it is also always possible to consult professors Google and Wikipedia for reference recommendations. Please clear your specific topic choices with me.

The topics below directly extend those of the class. Students comfortable with the notion of manifold have many more options and are welcome to discuss them with me; I may add some such options below later.

- **Total curvature of space curves** Kühnel's Theorem 2.34 is a result of Fenchel: the total curvature of a space curve in $\mathbb{R}^3$ is at least 2π. J. Milnor's very first paper [7] proves a follow-up conjecture of Borsuk: if the curve is *knotted* then its curvature is greater than 4π. Either this or the next topic could be interesting for a student interested in knots.

- **Möbius energy of links** This is an another integral invariant of knots (or links) in $\mathbb{R}^3$ with a number of nice properties, see [2] and the follow-up [4]. A conjecture made in [2] was very recently proved by Agol–Marques–Neves, see [6] or `http://arxiv.org/abs/1205.0825`

- **The Willmore conjecture** See Kühnel's Exercise 3.16. This was subsequently proved by Marques–Neves, who have a nice survey article at `http://arxiv.org/abs/1409.7664`. The article contains a lot of further references to interesting related mathematics.

- **Minimal surfaces** A vast and beautiful subject area. The Wikipedia page has a nice historical overview and many references. Sub-topics include:
  - **Weierstrass representation** See Kühnel Section 3D, in particular Corollary 3.36. The idea here is that a minimal surface may be described locally as the real part of a complex-analytic function depending on parameters which may be (somewhat) freely chosen.
  - **Soap films/Plateau's problem** Plateau's problem asks whether, for a given boundary curve γ, there exists a least-area surface with boundary γ; such a surface must in particular be minimal. Physical principles mandate that a soap film suspended from a wire be such a surface; the shape of the wire prescribes the topology of the surface. For a survey of some developments in minimal surfaces related to this problem see `http://arxiv.org/pdf/1308.3325v2.pdf`; for a computational approach to some examples see eg. `http://www.jimrolf.com/explorationsInComplexVariables/bookChapters/Ch2.pdf`
  - **Costa's minimal surface** The first counterexample to a long-standing conjecture that the plane, catenoid and helicoid are the only complete embedded minimal surfaces with "finite topology". Still looking for a good basic reference, but see the Wikipedia page or Wolfram mathworld.
  - **Triply periodic surfaces** Surfaces invariant under a group of Euclidean isometries that has a compact fundamental domain in $\mathbb{R}^3$. In contrast to

Costa's surface, these all have "infinite topology". Their classification is an open problem, and in many cases it is easier to numerically describe examples than prove they are minimal. See the Wikipedia page for some references; for others and some really lovely pictures see
`http://www.susqu.edu/brakke/evolver/examples/periodic/periodic.html`
Brakke's home page (just remove the stuff after "brakke" above) has links to a number of other interesting topics.

- **Weingarten surfaces** Their principal curvatures satisfy a relation; see Definition 3.26 in Kühnel. There is a nice overview of the study of Weingarten surfaces in the intro to [5] (coauthored by "the very Kühnel"); the rest of this paper should be reasonably accessible to a student familiar with the classification of surfaces.

- **Geometric flows** For the analytically-minded among us. The idea is to "improve" a hypersurface element by evolving it over time according to a curvature-determined differential equation. Standard concerns include short-time existence and uniqueness of solutions, and long-term behavior. Among the classic examples are:
  - **Mean curvature flow** A family f_t of hypersurface elements on a fixed domain U evolves by mean curvature flow if $\frac{\partial}{\partial t} f(\mathbf{u}) = H_t(\mathbf{u})\nu_t(\mathbf{u})$; where $H_t(\mathbf{u})$ is the mean curvature of f_t at $\mathbf{u}$ and $\nu_t(\mathbf{u})$ is the normal vector. It locally decreases area, see above Theorem 3.28 in Kühnel. Lots of current research interest in this flow, see eg. `http://arxiv.org/pdf/1102.1411v1.pdf` (I'll try to find a more accessible reference).
  - **Curve-shortening flow** The 1-dimensional case of mean curvature flow. A nice and (at least initially) very accessible overview is at
    `http://smp.uq.edu.au/sites/smp.uq.edu.au/files/GeometricFlowsBook_Ch2.pdf`

  Other well-studied geometric flows of hypersurfaces include the Gauss curvature flow (see eg. [1]) and the harmonic map heat flow [3]. The Ricci flow on a Riemannian manifold, recently used by Perelman to prove the 3-dimensional Poincaré conjecture (see the survey article [8]), is an intrinsic analog of the mean curvature flow.

References

[1] Ben Andrews. Gauss curvature flow: the fate of the rolling stones. *Invent. Math.*, 138(1):151–161, 1999.
[2] Steve Bryson, Michael H. Freedman, Zheng-Xu He, and Zhenghan Wang. Möbius invariance of knot energy. *Bull. Amer. Math. Soc. (N.S.)*, 28(1):99–103, 1993.
[3] James Eells, Jr. and J. H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86:109–160, 1964.
[4] Michael H. Freedman, Zheng-Xu He, and Zhenghan Wang. Möbius energy of knots and unknots. *Ann. of Math. (2)*, 139(1):1–50, 1994.
[5] Wolfgang Kühnel and Michael Steller. On closed Weingarten surfaces. *Monatsh. Math.*, 146(2):113–126, 2005.
[6] Fernando C. Marques and André Neves. Min-max theory, Willmore conjecture and the energy of links. *Bull. Braz. Math. Soc. (N.S.)*, 44(4):681–707, 2013.
[7] J. W. Milnor. On the total curvature of knots. *Ann. of Math. (2)*, 52:248–257, 1950.
[8] John W. Morgan. Recent progress on the Poincaré conjecture and the classification of 3-manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 42(1):57–78 (electronic), 2005.