DIFFERENTIAL GEOMETRY 2
POSSIBLE FINAL TALK TOPICS

The list below may grow as things occur to me, and I will insert good references as I discover them (published papers may be found on mathscinet: `www.ams.org/mathscinet`). Of course a Google search can turn up other references, and Wikipedia sometimes has helpful entries (with more references). Do not feel bound by the list, particularly if you are more advanced, but please do clear your specific choice of topic with me.

Some topics below are more in the purview of differential topology than differential geometry, strictly speaking, and the first is probably the most elementary of all, but it still has some subtle aspects. Many topics below have significantly more material than can be discussed in a single 50 min talk. You can discuss we me about what to cover, and/or team up with a classmate to give a series of talks covering different aspects of the same material.

0.1. **The classification of one- and two-manifolds.** Both of these are "classical" results that were known in the 1800's at some level of rigor. Guillemin and Pollack's *Differential Topology* proves the classification of smooth one-manifolds in an appendix. There is a completely elementary proof of the classification of topological one-manifolds in [5] (google "classification of one-manifolds").

The classification of two-manifolds is proved in eg. Massey's *Algebraic Topology.* The method there uses triangulations and relies on a result, usually attributed to Radó, that every topological surface is triangulable. In his recent preprint [7], A. Hatcher gave a nice short proof of the related result that every topological surface admits a smooth structure. In [4], Francis–Weeks outline "Conway's ZIP proof" (google it), a proof of the classification of surfaces that still relies on triangulations but has (according to Conway) Zero Irrelevancy. That is, it's best possible. Check out [4] for the beautiful pictures drawn by Francis.

0.2. **Morse Theory.** For a differentiable manifold M, a smooth function $f\colon M \to \mathbb{R}$ is *Morse* if at every *critical point* x, where $Df_x \equiv 0$, the *Hessian matrix* $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)$ is non-singular. The existence, indeed the ubiquity, of Morse functions follows from **Sard's Theorem**, a very important basic result in differential topology described in Guillemin and Pollack §1.7. Morse functions are a very useful tool in understanding the topology of smooth manifolds, see Milnor's *Morse Theory.*

0.3. **The space form problem.** This is the problem of classifying all manifolds that admit constant-curvature metrics. We will prove in class that each n-dimensional space form X is of the form $\widetilde{X}/\Gamma$, where $\widetilde{X}$ is one of the *model spaces* S^n, $\mathbb{R}^n$, or $\mathbb{H}^n$, equipped with a metric g of constant curvature greater than, equal to, or less than 0, respectively, and Γ is a discrete group of isometries of $(\widetilde{X}, g)$ acting freely on $\widetilde{X}$.

The space form problem has a different flavor than the differential geometry we've discussed in class, more like classical Euclidean geometry. It naturally divides into three sub-problems, on classifying:

- **spherical manifolds**, for which $\widetilde{X} = S^n$;
- **Euclidean manifolds**, for which $\widetilde{X} = \mathbb{R}^n$; or
- **hyperbolic manifolds**, for which $\widetilde{X} = \mathbb{H}^n$.

One could easily spend an entire talk discussing only one of these sub-problems; for instance by talking about some particulars distinguishing it from the others, surveying what is currently known, and then discussing the dimension-two and three subcases. A little algebraic topology knowledge is helpful for this problem. Kühnel's Section 7C is one reference.

0.4. **Exotic spheres.** An *exotic sphere* is a smooth manifold homeomorphic but not diffeomorphic to the "standard sphere" $S^n \subset \mathbb{R}^n$. Every two- or three-dimensional topological manifold admits a smooth structure that is unique up to diffeomorphism. This no longer holds true in general for 4-manifolds, but it is unknown for the special case of S^4; that it does is the **smooth Poincaré conjecture**.

On the other hand it is known that there are no exotic 5- or 6-dimensional spheres, but there are several in dimension 7. For an overview on exotic spheres, which were first discovered by Milnor [8], see the nice Wikipedia entry.

0.5. **Local-to-global principles.** A local-to-global principle is a way of translating information that is known about a space X in a neighborhood of each point into global information about X. Here are a couple very famous examples:

- The **Sphere Theorem** asserts that every Riemannian manifold of "quarter-pinched" sectional curvatures is homeomorphic to a spherical space form. Brendle–Schoen recently showed that "homeomorphic" can be replaced by "diffeomorphic" in the conclusion here, in fact with somewhat weaker hypotheses, see [1].
- The **Cartan–Hadamard Theorem**: If X is a Riemannian n-manifold with sectional curvature everywhere less than 0 then the universal cover of X is diffeomorphic to $\mathbb{R}^n$. This has some important topological consequences, for instance that the fundamental group is infinite if X is compact.

Chapter 11 of John M. Lee's *Riemannian Manifolds* has a good overview of these and other local-to-global results, with proofs in some cases (including of Cartan–Hadamard).

0.6. **Lie groups.** A *Lie group* G is a smooth manifold that that is also a group, such that the group operations determine smooth functions. In particular, G acts on itself by left- or right-multiplication, and much of the rich structure theory of Lie groups is developed by analyzing objects that are invariant under one or both of these actions. Warner's Chapter 3 has a concise introduction to Lie groups using only differential topology.

Along the lines of differential geometry, Lee has a sequence of problems constraining various aspects of (mostly) bi-invariant Riemannian metrics on Lie groups, those invariant under left- and right-translation; see Problems 3-10–3-12, 5-11, 7-5 and 8-14 there. Milnor has a nice survey article on curvatures of left-invariant metrics [10].

A **homogeneous space** of a Lie group G is a manifold M with a transitive G-action by smooth functions. Onischik–Vinberg assert that "Homogeneous spaces are the most interesting and important objects of geometry" [12, p. 12]. So there. The constant curvature spaces are examples of homogeneous spaces, but there are many more. For instance, there are 9 three-dimensional homogeneous spaces, and the Geometrization Theorem asserts that every closed three-manifold has a canonical topological decomposition into pieces, each locally modeled on one of them.

There is a 1-1 correspondence between G-invariant Riemannian metrics on a homogeneous space M of G and certain left-invariant metrics on G itself, which can be understood very naturally with the aid of some of the basic objects of Lie theory (eg. Lie subgroups, the adjoint action). Establishing this could make for a nice talk.

A **symmetric space** of G is a homogeneous space M such that the stabilizer in G of any $x \in M$ satisfies an additional criterion whose significance is not really apparent at first glance. But the symmetric spaces are particularly important among homogeneous spaces and include for instance the constant-curvature spaces. The Cartan–Ambrose–Hicks Theorem characterizes Riemannian symmetric spaces via the condition $\nabla R = 0$. See Exercise 8 and Theorem 2.1 of Ch. 8 in do Carmo's *Riemannian Geometry*.

0.7. **Contact and symplectic manifolds.** A *contact* form on a $(2n+1)$-dimensional manifold M is a nowhere-vanishing one-form α with the property that $\alpha \wedge (d\alpha)^n$ is also non-zero. Equivalently, a *contact structure* on M is a *totally non-integrable distribution* of hyperplanes $H_p \subset T_pM$ (take $H_p = \ker \alpha_p$). A *symplectic* form ω on a $(2n)$-dimensional manifold M is a closed, 2-form on M with $\omega^n \neq 0$.

The studies of compact and symplectic manifolds share many similarities. For instance, in both cases there is "no local geometry": every contact form α is locally equivalent to:

$$dz + \sum_{i=1}^{n} x^i \, dy^i$$

in some chart $(z, x^1, y^1, \ldots, x^n, y^n)$, and every symplectic form has a similar description. So most questions are of a global nature, on the existence of contact or symplectic forms on arbitrary manifolds, the classification of nice submanifolds, and so on.

See the freely downloadable book *Lectures on Symplectic Geometry* by Ana Cannas da Silva.

0.8. **Bundles and connections.** A *vector bundle* $\pi\colon E \to M$ over a smooth manifold M is a smooth manifold E which is the disjoint union of a collection of vector spaces E_p, one for each $p \in M$, with the following properties:

- π is smooth, where $\pi(\mathbf{v}) = p$ for all $\mathbf{v} \in E_p$; and
- each $p \in M$ has a neighborhood U in M such that $\pi^{-1}(U)$ is *locally trivial*; ie, $\pi^{-1}(U) \cong U \times \mathbb{R}^n$ for some n, by a map that restricts to a vector space isomorphism on any fiber $\pi^{-1}(q)$, $q \in U$.

You have seen the *tangent bundle* in a homework exercise, where $E_p = T_pM$; the *cotangent bundle* mentioned above has as its fibers the dual spaces T_pM^*. Other naturally occurring bundles collect forms and tensors on the tangent spaces.

A *section* of a bundle $\pi\colon E \to M$ is a smooth map $s\colon M \to E$ such that $\pi \circ s$ is the identity map on M. For instance, a vector field on M is a smooth section of the tangent bundle, and a one-form is a smooth section of the cotangent bundle. We studied the Riemannian connection ∇ in class, which in this context it is natural to take as an operator $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the space of smooth vector fields on M. For an arbitrary vector bundle $E \to M$, if $\mathfrak{E}(M)$ is the space of smooth sections of E we say a *connection* on E is an operator $\mathfrak{X}(M) \times \mathfrak{E}(M) \to \mathfrak{E}(M)$ with the linearity properties of the Riemannian connection but not the others.

A connection is determined by a set of one-forms called the *connection one-forms*, and it determines a *curvature 2-form*, see eg. Kühnel's chapter 4 or [9]. It determines a notion of parallel transport and hence also of holonomy.

0.9. **Complex and Kähler manifolds.** Again see *Lectures on Symplectic Geometry*.

0.10. **The Poincaré conjecture and Geometrization in dimension three.** The Poincaré conjecture is an essentially algebraic-topological assertion: every closed n-manifold homotopy-equivalent to the n-dimensional sphere is homeomorphic to it. This was proven by Smale for $n \geq 5$, and by Freedman for $n = 4$, in both cases using essentially algebraic topological techniques. The assertion for $n = 2$ is a consequence of the classification of surfaces.

This leaves the case $n = 3$, which remained open for longer, and was given a geometric context as one part of W. Thurston's Geometrization Conjecture for 3-manifolds: that every 3-manifold has a canonical topological decomposition into pieces, each of which is locally modeled on one of eight 3-dimensional homogeneous spaces. G. Perelman proved the Geometrization Conjecture in 2003 using Ricci flow. Check out the Wikipedia page or J. Morgan's survey article [11], which is freely available online.

0.11. **Einstein spaces.** See Kühnel's Chapter 8.

0.12. **Ricci flow.** This is a *geometric evolution equation*: starting with a Riemannian metric g_0 on a smooth manifold M, one prescribes a family of metrics $\{g_t\}_{t\geq 0}$ on M using a differential equation. Since g is a symmetric $(0,2)$-tensor, its time derivative is as well. In the case of Ricci flow, this derivative is given by the (scaled) Ricci tensor:

$$\frac{\partial}{\partial t} g_t = -2\mathrm{Ric}_t,$$

where Ric_t is the Ricci tensor of the metric g_t. Writing this down in local coordinates yields a complicated-looking second-order system of PDE in the coefficients of g.

The usual PDE concern of well-posedness (short-time existence and uniqueness of solutions) applies here. It was originally proved by R. Hamilton in [6], where he introduced the Ricci flow; later a simplified proof was given by D. DeTurck (via the "DeTurck trick") [3]. Having proved this, the main goal is to understand what Ricci flow does to the geometry of the manifold. This is currently an active field of study, but quite a bit is now known thanks to work of many authors. See me for specific references, but one general introduction is [2].

Some special cases are not too hard to analyze. For instance, Einstein metrics evolve by scaling, and there is a more general class of metrics called *solitons* which evolve by diffeomorphism. Such metrics are essentially fixed points of the flow and so occupy a special place in the theory. Of course it is important to note that not every manifold admits an Einstein or soliton metric, and an important problem is to classify those that do.

In a different direction, the Ricci flow on a homogeneous space reduces to a system of ODE using the Lie group action (an observation of Milnor, natch). This is a good source of examples that can be analyzed directly but are not completely trivial.

References

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