TOPOLOGY 2, HOMEWORK 10
PROBLEM 1 SOLUTION

Problem: For a fixed space X and $x_0 \in X$, we showed in class that a map $f\colon (I^n, \partial I^n) \to (X, x_0)$ induces a map $\bar{f}\colon (S^n, \mathbf{s}_0) \to (X, x_0)$, where $\mathbf{s}_0 = (0, \dots, 0, 1)$, obtained by going right-to-left on the bottom line of the diagram below.

$$\begin{tikzcd}
 & I^n \arrow[r, "\phi"] \arrow[dl, "f"'] \arrow[d] & \mathbb{D}^n \arrow[d] \arrow[dr] & \\
X & I^n/\partial I^n \arrow[l, dotted] \arrow[r, "\approx"] & \mathbb{D}^n/\partial\mathbb{D}^n \arrow[r, "\approx"] & S^n
\end{tikzcd}$$

For two such maps f and g, show that the map $\overline{f+g}$ induced in this way is the same as $(\bar{f} \vee \bar{g}) \circ c$, where $c\colon S^n \to S^n/(\{0\} \times S^{n-1})$ is the quotient map and $S^n/(\{0\} \times S^{n-1})$ is homeomorphically identified with $S^n \vee S^n$.

Solution: Let us recall the definitions of maps $\phi\colon I^n \to \mathbb{D}^n$ and $\psi\colon \mathbb{D}^n \to S^n$ that induce the homeomorphisms of the bottom line above:

$$\phi(\mathbf{s}) = 2 \max_i \{|s_i - 1/2|\} \frac{\mathbf{s} - \mathbf{1/2}}{\|\mathbf{s} - \mathbf{1/2}\|} \qquad \mathbf{s} = (s_1, \dots, s_n) \in I^n$$

$$\psi(\mathbf{x}) = \left(2\sqrt{1 - \|\mathbf{x}\|^2}\, \mathbf{x}, 2\|\mathbf{x}\|^2 - 1\right) \qquad \mathbf{x} \in \mathbb{D}^n$$

Above, $\mathbf{1/2} = (1/2, \dots, 1/2) \in I^n$. Let $\Phi = \psi \circ \phi$. Note that Φ takes ∂I^n to $\mathbf{e}_{n+1} = (0, \dots, 0, 1) \in S^n$, which we will take as a base point.

We first observe that $\Phi(\{1/2\} \times I^{n-1}) = \{0\} \times S^{n-1}$, so since $f+g$ maps $\{1/2\} \times I^{n-1}$ to the basepoint x_0 it induces a map from the further quotient $S^n/(\{0\} \times S^{n-1})$ to X. For lack of a better term we will call this map $\overline{f+g}^*$. It satisfies $\overline{f+g}^* \circ c = \overline{f+g}$, where $c\colon S^n \to S^n/(\{0\} \times S^{n-1})$ is the quotient map. We will show that $\overline{f+g}^*$ is homotopic to $\bar{f} \vee \bar{g}$ relative to $\{0\} \times S^{n-1}$, which is identified with the wedge point under any homeomorphism $S^n/(\{0\} \times S^{n-1}) \to S^n \vee S^n$, and hence conclude that $\overline{f+g}$ is homotopic to $(\bar{f} \vee \bar{g}) \circ c$, relative to $\mathbf{e}_{n+1}$.

Denote as S^n_l the copy of S^n in the wedge sum which is obtained from points $\mathbf{x} \in S^n/(\{0\} \times S^{n-1})$ with $x_1 \le 0$, and let S^n_r be the other copy. We will show that the restriction of $\overline{f+g}^*$ to S^n_l is homotopic to $\bar{f}$ by describing the latter as obtained from the former by a reparametrization homotopic (rel $\mathbf{e}_{n+1}$) to the identity map. A similar description of $\bar{g}$ will establish the result. The key claim is:

Claim. Let $X = [0, 1/2] \times I^{n-1}$. The restriction of $c \circ \Phi$ to X is homotopic to $\Phi \circ 2s_1$ relative to ∂X, where "$2s_1$" refers to the map sending $(s_1, \dots, s_n)$ to $(2s_1, s_2, \dots, s_n)$, and both maps above have target S^n_l. Similarly, the restriction of $c \circ \Phi$ to $[1/2, 1] \times I^{n-1}$ is homotopic to $\Phi \circ (2s_1 - 1)$ rel boundary, where "$2s_1 - 1$" is analogously defined and both of these maps target S^n_r.

Assuming the claim for the moment, we now prove the result. For $\mathbf{y}_0 = c \circ \Phi(\mathbf{s}) \in S^n_l$, $\overline{f+g}^*(\mathbf{y}_0) = (f+g)(\mathbf{s}) = f(2s_1, s_2, \ldots, s_n)$, where $\mathbf{s} = (s_1, \ldots, s_n) \in [0, 1/2] \times I^{n-1}$. This is also $\bar{f}(\mathbf{y}_1)$, where $\mathbf{y}_1 = \Phi \circ 2s_1(\mathbf{s})$. So on $[0, 1/2] \times I^{n-1}$ we have

$$\overline{f+g}^* \circ (c \circ \Phi) = f \circ 2s_1 = \bar{f} \circ (\Phi \circ 2s_1)$$

Each of $c \circ \Phi$ and $\Phi \circ 2s_1$ maps the entire boundary of $X = [0, 1/2] \times I^{n-1}$ to $\mathbf{e}_{n+1}$ and is one-one on the interior, so it induces a homeomorphism $X/\partial X \to S^n_l$. Denote these maps by $\overline{c \circ \Phi}$ and $\overline{\Phi \circ 2s_1}$, respectively. Then

$$\bar{f} = \overline{f+g}^* \circ (\overline{c \circ \Phi}) \circ (\overline{\Phi \circ 2s_1})^{-1}.$$

A homotopy $H\colon X \times I \to S^n_l$ from $c \circ \Phi$ to $\Phi \circ 2s_1$ relative to ∂X induces a homotopy $\overline{H}\colon (X/\partial X) \times I \to S^n_l$ between induced maps, so composing $\overline{f+g}^*$ with $\overline{H}\left((\overline{\Phi \circ 2s_1})^{-1}(\mathbf{y}), t\right)$ gives one from $\bar{f}$ to the restriction of $\overline{f+g}^*$ to S^n_l.

A similar construction gives a homotopy on S^n_r from $\bar{g}$ to the restriction of $\overline{f+g}^*$. These homotopies fix the wedge point, since it is the image of ∂X under $c \circ \Phi$ and $\Phi \circ 2s_1$, and similarly for $c \circ \Phi$ and $\Phi \circ (2s_1 - 1)$. Therefore $\overline{f+g}^*$ is homotopic to $\bar{f} \vee \bar{g}$ relative to the wedge point, so $\overline{f+g}$ is homotopic to $(\bar{f} \vee \bar{g}) \circ c$ rel $\mathbf{e}_{n+1}$.

It remains to prove the claim. We must first define the quotient map $c\colon S^n \to S^n \vee S^n$ that identifies $S^n \vee S^n$ with $S^n/(\{0\} \times S^{n-1})$. For $\mathbf{y} = (y_1, \ldots, y_{n+1})$, let:

$$c(\mathbf{y}) = \begin{cases} \psi(-y_{n+1}, y_2, \ldots, y_n)_l & y_1 \leq 0 \\ \psi(y_{n+1}, y_2, \ldots, y_n)_r & y_1 \geq 0 \end{cases}$$

Here for $\mathbf{x} = (y_2, \ldots, y_{n+1}) \in \mathbb{D}^n$, $\psi(\mathbf{x})_l$ lies in S^n_l and $\psi(\mathbf{x})_r$ lies in S^n_r. These spheres are wedged along $\mathbf{e}_{n+1}$. Note that c takes the entire equator $\{0\} \times S^{n-1}$ to the wedge point, so it is continuous, and it is one-to-one off of the equator. Therefore c induces a homeomorphism $S^n/(\{0\} \times S^{n-1}) \to S^n \vee S^n$ as desired.

Remark. It may seem more natural to define $c(\mathbf{y})$ as $\psi(y_2, \ldots, y_{n+1})$ (note that since c crushes $\{0\} \times S^{n+1}$ it may as well factor through the projection that forgets the first entry). But up to multiplication by a positive scalar, this reshuffles the first n entries of $\mathbf{y}$ by a cyclic permutation which is orientation-reversing for some n, so for the claim to hold it is necessary to correct this.

Proof of claim. For $\mathbf{s} = (s_1, \ldots, s_n) \in X - \partial X$ let $\mathbf{z} = (z_1, \ldots, z_{n+1}) = c \circ \Phi(\mathbf{s}) \in S^n_l$. The main point of the proof is to show that $\mathbf{z} \neq -(\Phi \circ 2s_1)(\mathbf{s})$. This being the case, the straight-line homotopy in $\mathbb{R}^{n+1}$ from $\mathbf{z}$ to $(\Phi \circ 2s_1)(\mathbf{s})$ yields a homotopy in S^n_l after radial projection. Moreover, since each of these maps sends ∂X to $\mathbf{e}_{n+1}$, the resulting homotopy is relative to ∂X.

We begin by noting that there is a positive constant λ such that $z_i = \lambda(s_i - 1/2)$ for each $i \in \{2, \ldots, n\}$. For such i, the ith entry of $(\Phi \circ 2s_1)(\mathbf{s})$ is also a positive multiple of $s_i - 1/2$, so if $\mathbf{z} = -(\Phi \circ 2s_1)(\mathbf{s})$ then $s_i = 1/2$ for each such i and $\mathbf{s} = (t, 1/2, \ldots, 1/2)$ for some $t \in [0, 1/2]$. For such $\mathbf{s}$, direct computation reveals:

$$\begin{aligned}(\Phi \circ 2s_1)(\mathbf{s}) &= \left(4\sqrt{2t(1-2t)}(4t-1), 0, \ldots, 0, 2(4t-1)^2 - 1\right) \\ z_1 &= 8\sqrt{t(1-t)}(1-2t)(8t - 8t^2 - 1) \\ z_{n+1} &= 2(8t^2 - 8t + 1)^2 - 1\end{aligned}$$

Note that both z_1 and the first entry of $(\Phi \circ 2s_1)(\mathbf{s})$ are 0 at $t = 0$ and $1/2$, negative for $t > 0$ near 0, and positive for $t < 1/2$ near $1/2$. Solving for their other zeros, we find that their signs disagree only for t in the interval $\left(\frac{2-\sqrt{2}}{4}, \frac{1}{4}\right)$. But the last entry of $\Phi \circ 2s_1)(\mathbf{s})$ is negative on the interval $\left(\frac{2-\sqrt{2}}{8}, \frac{2+\sqrt{2}}{8}\right)$, and z_{n+1} is negative on $\left(\frac{2-\sqrt{2+\sqrt{2}}}{4}, \frac{2-\sqrt{2-\sqrt{2}}}{4}\right)$. The intersection of these two intervals contains $\left(\frac{2-\sqrt{2}}{4}, \frac{1}{4}\right)$, so $\mathbf{z} \neq -(\Phi \circ 2s_1)(\mathbf{s})$ for any $\mathbf{s} \in X$.

A similar argument shows that $c \circ \Phi$ is homotopic to $\Phi \circ (2s_1 - 1)$ on $[1/2, 1] \times I^{n-1}$. In particular, for $\mathbf{s} = [t, 1/2, \ldots, 1/2]$ with $1/2 \leq t \leq 1$ we have:

$$\begin{aligned}
(\Phi \circ (2s_1 - 1))(\mathbf{s}) &= \left(4\sqrt{(2t-1)(2-2t)}(4t-3), 0, \ldots, 0, 2(4t-3)^2 - 1\right) \\
z_1 &= 8\sqrt{t(1-t)}(2t-1)(8t^2 - 8t + 1) \\
z_{n+1} &= 2(8t^2 - 8t + 1)^2 - 1
\end{aligned}$$

Here $\mathbf{z} = (z_1, 0, \ldots, 0, z_{n+1}) = (c \circ \Phi)(\mathbf{s})$. $\square$