PRESENTATIONS OF ABELIAN GROUPS

Lemma. *A group G with generating set $\mathcal{S}$ is abelian if and only if $[g,h]=1$ for all $g,h\in\mathcal{S}$, where $[g,h]=ghg^{-1}h^{-1}$ by definition, and "1" is the identity of G.*

Proof. If G is abelian then $[g,h]=1$ for all g and h in G, so in particular also for all g and h in $\mathcal{S}$. Now suppose $[g,h]=1$ for all $g,h\in\mathcal{S}$. Tthat this holds for all $g,h\in G$ is equivalent to:

Claim. For all words v and w in the free group $F(\mathcal{S})$, $[v,w]\in\langle\langle\{[g,h]\,|\,[g,h]\in\mathcal{S}\}\rangle\rangle$.

We will prove the claim by induction on *word length*, which is defined for a reduced word

$$w=g_1^{k_1}g_2^{k_2}\cdots g_n^{k_n}\in F(\mathcal{S})-\{\emptyset\}$$

as $|k_1|+|k_2|+\ldots+|k_n|$. (Here $g_i\in\mathcal{S}$ and $k_i\in\mathbb{Z}-\{0\}$ for all i, and $g_{i+1}\neq g_i$ for each $i<n$.) The base case follows by hypothesis and the observation that for all $g,h\in\mathcal{S}$:

$$[g^{-1},h]=g^{-1}hgh^{-1}=g^{-1}(hgh^{-1}g^{-1})g=g[h,g]g^{-1}\in\langle\langle\mathcal{S}\rangle\rangle$$

For the induction step, suppose the claim holds for all words of length at most n, and for $g,h\in\mathcal{S}$ and words v and w of length at most n, consider the commutator $[gv,hw]$:

$$\begin{aligned}[gv,hw]&=gvhw(gv)^{-1}(hw)^{-1}=gv\,hw\,v^{-1}g^{-1}\,w^{-1}h^{-1}\\&=gvh(v^{-1}v)wv^{-1}(w^{-1}w)g^{-1}w^{-1}h^{-1}=g\,vhv^{-1}\,[v,w]\,wg^{-1}w^{-1}\,h^{-1}\\&=gh[h^{-1},v]\,[v,w]\,[w,g^{-1}]\,g^{-1}h^{-1}\\&=gh[h^{-1},v]\,[v,w]\,[w,g^{-1}]\,[g^{-1},h^{-1}]\,h^{-1}g^{-1}\end{aligned}$$

All of the commutators above are of elements with word length at most n, so by the inductive hypothesis they lie in $\langle\langle\mathcal{S}\rangle\rangle$. Therefore so does $[gv,hw]$. Since an arbitrary word of length $n+1$ is of the form gv for some $g\in\mathcal{S}$ and v of length n, the claim and hence the lemma is proved by induction. □