ON THE DOUBLED TETRUS

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Abstract. The "tetrus" is a member of a family of hyperbolic 3-manifolds with totally geodesic boundary, described by Paoluzzi–Zimmerman, which also contains W.P. Thurston's "tripus." Each member of this family has a branched cover to B^3 over a certain tangle T. This map on the tripus has degree three, and on the tetrus degree four. We describe a cover of the double of the tetrus, itself a double across a closed surface, which fibers over the circle.

This paper describes some features of a fibered cover of a certain 3-manifold, related to a family of compact hyperbolic 3-manifolds $M_{n,k}$ with totally geodesic boundary defined by Paoluzzi-Zimmermann [20]. A well known member of this family is $M_{3,1}$, Thurston's "tripus" [25, Ch. 3]. Here we consider the "tetrus" $M_{4,1}$ (thanks to Richard Kent for naming suggestions).

If M is an oriented manifold with boundary, let $\overline{M}$ be a copy of M with orientation reversed, and define the *double* of M to be $DM = M \cup_\partial \overline{M}$, where the gluing isometry $\partial M \to \partial\overline{M}$ is induced by the identity map.

Theorem 0.1. *There is a cover $p\colon D\widetilde{M} \to DM_{4,1}$ of degree 6, where $D\widetilde{M}$ is a double across the closed surface $p^{-1}(\partial M_{4,1})$ of genus 13, and $D\widetilde{M}$ fibers over S^1 with fiber $\widetilde{F}$, a closed surface of genus 19.*

The manifold $D\widetilde{M}$ above is the first hyperbolic 3-manifold which we know to be both fibered and a double across a closed surface. Non-hyperbolic fibered doubles are easily constructed, for instance by doubling the exterior of a fibered knot across its boundary torus, but producing a fibered hyperbolic double is a more subtle problem. In such a manifold the doubling surface — necessarily with genus at least 2 — does not itself admit a fibering and must thus have points of tangency with any fibering, which in particular cannot be invariant under the doubling involution.

The strategy of proof for Theorem 0.1 is motivated by the fact that the doubled tetrus branched covers S^3, branched over the link L of Figure 1, which is the Montesinos link $L(1/3, 1/2, -1/2, -1/3)$. Thurston observed that given a branched cover $M_n \to M$, branched over a link L, virtual fiberings of M transverse to the preimage of L may be pulled back to virtual fiberings of M_n. This is also used in [4], which is where we encountered it, and [3]. We record a version of this observation as Proposition 1.2, and apply it here with Proposition 1.3 to prove Theorem 0.1.

When $M' = F \times S^1$, where F is a closed surface, and L is a link contained in a disjoint union of fibers, Proposition 1.3 produces new fiberings of M' transverse to L under certain circumstances. We state and prove Proposition 1.3 in Section 1, along with Proposition 1.2. In Section 2 we describe the manifolds $M_{n,k}$ constructed in [20], and their branched covers to the 3-ball. The branched cover $DM_{4,1} \to S^3$, which results from doubling, factors through a double branched cover $DM_2 \to$

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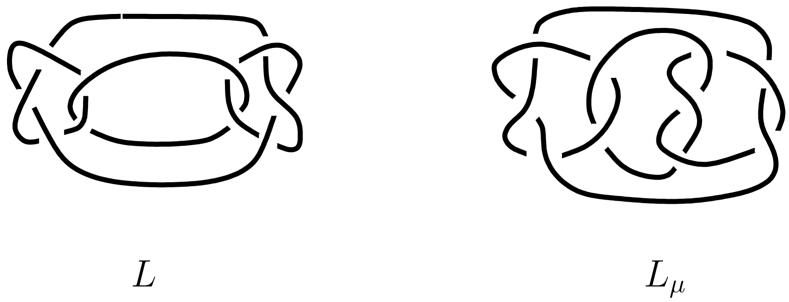

FIGURE 1. $DM_{n,k}$ and $D_\mu M_{n,k}$ n–fold cover S^3, branched over L and L_μ.

S^3, branched over L. The double branched cover of S^3 over a Montesinos link is well known to have the structure of a Seifert fibered space ([17], cf. [7]). In Section 3, we describe the Seifert fibered structure on DM_2 and construct a cover $p'\colon DM' \to DM_2$ which satisfies the hypotheses of Proposition 1.3. An application of Proposition 1.2 completes the proof of Theorem 0.1.

Steven Boyer and Xingru Zhang previously obtained results, eventually published in [3], about virtual fiberings of Montesinos links and their branched covers using a similar strategy. These imply that the doubled tetrus is virtually fibered; in particular, [3, Theorem 1.7] applies more generally than our independently discovered Proposition 1.3. An advantage of Proposition 1.3, when it applies, is that it produces an explicit description of a fiber surface, which we use in Theorem 0.1 to obtain the extra information about the genus of $\widetilde{F}$.

Remark. Our methods apply without significant alteration to the manifolds $DM_{2n,1}$ for each $n \geq 2$, in each case producing a sixfold fibered cover which is a double. See the remarks below Lemma 3.3 and the proof of Theorem 0.1. We have chosen to focus on the tetrus because of its peculiarity described in Proposition 0.3 below.

It is well known that $M_{n,k}$ has a minimal–genus Heegaard splitting of genus n, obtained by attaching a single one–handle to $\partial M_{n,k}$. (Ushijima classified such splittings in [26, Theorem 2.8].) Thus the tetrus $M_{4,1}$ has a Heegaard splitting with genus 4, yielding an amalgamated Heegaard splitting of $DM_{4,1}$ with genus 5. The preimage in $\widetilde{DM}$ gives the following corollary of Theorem 0.1.

Corollary 0.2. *$\widetilde{DM}$ has a weakly reducible Heegaard splitting of genus 25 associated to $p^{-1}(\partial M_{4,1})$, and one of genus 39 associated to $\widetilde{F}$.*

It would be interesting to know the minimal genus of a fiber surface for $\widetilde{DM}$, for the above discussion shows that if this is greater than twelve, the minimal genus Heegaard splitting of $\widetilde{DM}$ is not associated to a fibering. Such examples are nongeneric, according to work of Souto [23, Theorem 6.2] and Biringer [6]. See also [5] for a survey of results about degeneration.

A *twisted double* $D_\mu M_{n,k}$ is obtained by gluing $M_{n,k}$ to its mirror image via an isometry μ of the boundary, the lift to $M_{n,k}$ of the mutation producing the link L_μ of Figure 1. Our original motivation for proving Theorem 0.1 was the dichotomy recorded below.

Proposition 0.3. *The doubled tetrus $DM_{4,1}$ is non-arithmetic, but $D_\mu M_{4,1}$ is arithmetic.*

We discuss the twisted doubling construction and sketch a proof of Proposition 0.3 in Section 4. Since the twisting map μ is an isometry, $D_\mu M_{4,1}$ contains a totally geodesic surface identical to the totally geodesic boundary of $M_{4,1}$, hence it follows from arithmeticity of $D_\mu M_{4,1}$ that this surface is arithmetic. This can also be discerned directly from a polyhedral decomposition. [13, Proposition 4.1] asserts:

Proposition (Long-Lubotzky-Reid)**.** *Let Γ be a Kleinian group of finite co-volume that contains an arithmetic Fuchsian subgroup. Then Γ contains a co-final nested family $\mathcal{L} = \{N_i\}$ of normal subgroups of finite index, such that Γ has Property τ with respect to $\mathcal{L}$.*

The corollary below follows immediately from this and arithmeticity of $\partial M_{4,1}$.

Corollary 0.4. *The doubled tetrus has a nested, cofinal family of regular covers with respect to which it has property τ.*

Work of Abert-Nikolov [1] concerning rank and Heegaard genus has drawn attention to virtually fibered manifolds which satisfy the conclusion of Corollary 0.4. By Theorem 0.1 and Proposition 0.3, the doubled tetrus is a non-arithmetic example of such manifolds. Other examples of this type can be derived from work of I. Agol [2]. These are commensurable to non-arithmetic right-angled reflection orbifolds. For example, the quotient of $\mathbb{H}^3$ by the group generated by reflections in the Löbell polyhedron $L(7)$ is non-arithmetic, contains totally geodesic surfaces, and has finite-degree cover which is a fibered manifold.

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1. Virtually fibering branched covers

We will be concerned in this paper with branched coverings. The *standard* k-fold cyclic branched covering of the disk D^2 to itself is the quotient map which identifies each point $z \in D^2 \subset \mathbb{C}$ with points of the form $ze^{2\pi i \frac{j}{k}}$, $0 \leq j < k$. For a 3-manifold M, an n-fold branched covering $q\colon M_n \to M$, branched over a link $L \subset M$, is characterized by the property that L has a closed regular neighborhood $\mathcal{N}(L)$, with *exterior* $\mathcal{E}(L) = \overline{M - \mathcal{N}(L)}$, such that

(1) On $\mathcal{E}_n \doteq q^{-1}(\mathcal{E}(L))$, q restricts to a genuine n-fold covering map.
(2) Each component V of $q^{-1}(\mathcal{N}(L))$ has a homeomorphism to $D^2 \times S^1$ so that $q|_V$ is the product of the standard k-fold branched cover with the identity map $S^1 \to S^1$, for some $k > 1$ dividing n.

Remark. It might be more accurate to allow the map in condition 2 to be the product of the k-fold branched covering of D^2 with a nontrivial covering of S^1 to itself. Since we will not encounter examples with this property here, we restrict our attention to the simpler setting.

Now suppose $p'\colon M' \to M$ is a genuine g-fold covering space, and let $\mathcal{E}' = (p')^{-1}(\mathcal{E}(L)) \subset M'$ be the associated cover of $\mathcal{E}(L)$. The group $\pi(L) \doteq \pi_1(\mathcal{E}(L))$ has subgroups Γ_n, and Γ', corresponding to the covers of $\mathcal{E}(L)$ by $\mathcal{E}_n$ and $\mathcal{E}'$, respectively. Below we record an elementary observation about Γ'.

Fact. Let $V \cong D^2 \times S^1$ be a component of $\mathcal{N}(L)$ and take $\mu = \partial D^2 \times \{y\} \subset \mathcal{E}(L)$ for some $y \in S^1$. If $[\mu]$ represents the homotopy class of μ in $\pi(L)$, then $[\mu] \in \Gamma'$.

This holds because $p'|_{\mathcal{E}'}$ extends to a covering map on M'; hence each component of the preimage of μ bounds a lift to M' of the inclusion map $D^2 \times \{y\} \hookrightarrow M$. We will call *meridians* curves of the form $\partial D^2 \times \{y\} \subset \partial\mathcal{E}(L)$. Take $\widetilde{\Gamma} = \Gamma_n \cap \Gamma'$, and let $\tilde{p}\colon \widetilde{\mathcal{E}} \to \mathcal{E}$ be the associated covering space, factoring through the restriction of q to $\mathcal{E}_n$ and of p' to $\mathcal{E}'$. We will say that $\widetilde{\mathcal{E}}$ *completes the diamond.*

Lemma 1.1. *Let $p'\colon M' \to M$ be a cover and $q_n\colon M_n \to M$ a branched cover, branched over a link $L \subset M$. Let $\mathcal{E}(L)$ be the exterior of L and $\mathcal{E}'$, $\mathcal{E}_n$ its covers associated to M' and M_n, respectively. The cover $\tilde{p}\colon \widetilde{\mathcal{E}} \to \mathcal{E}(L)$ that completes the diamond extends to $\tilde{p}\colon \widetilde{M} \to M$, where $\widetilde{M}$ is obtained by filling $\widetilde{\mathcal{E}}$ along preimages of meridians, such that the diagram below commutes.*

$$\begin{tikzcd}
\widetilde{M} \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, "\tilde{p}"] & M' \arrow[d, "p'"] \\
M_n \arrow[r, "q_n"] & M
\end{tikzcd}$$

Above, p is a covering map and q is a branched cover with the property that on each component $\widetilde{V}$ of $\tilde{p}^{-1}(\mathcal{N}(L))$, $\deg q|_{\widetilde{V}} = \deg q_n|_{p(\widetilde{V})}$.

Proof. Since $\widetilde{\mathcal{E}}$ corresponds to the subgroup $\widetilde{\Gamma} = \Gamma_n \cap \Gamma' < \pi(L)$, there are coverings $p\colon \widetilde{\mathcal{E}} \to \mathcal{E}_n$ and $q\colon \widetilde{\mathcal{E}} \to \mathcal{E}'$ such that $\tilde{p} = q_n \circ p = p' \circ q$.

Let $V \cong D^2 \times S^1$ be a component of $\mathcal{N}(L)$, let $\lambda = \{x\} \times S^1$ for some $x \in \partial D^2$, and let $\mu = \partial D^2 \times \{1\} \subset \mathcal{E}(L)$ be a meridian. Let V' be a component of the preimage of V in M', covering V g-to-1. The disk in V bounded by μ lifts to a disk in V' bounded by a component μ' of the preimage of μ under p', with the property that a component λ' of the preimage of λ intersects it once. Choosing a homeomorphism of V' with $D^2 \times S^1$ so that $\mu' = \partial D^2 \times \{1\}$ and $\lambda' = \{x\} \times S^1$, we find that p' is modeled on V' by the product of the identity map with the g-fold cover $S^1 \to S^1$ given by $w \mapsto w^g$.

On the other hand, since $M_n \to M$ is a branched cover, q_n restricts on each component of the preimage of V to the product of the k-fold branched cover $D^2 \to D^2$ with the identity map on S^1. In particular, q_n restricts to a homeomorphism on each component of the preimage of λ.

Now let $T = \partial V \subset \partial\mathcal{E}$, and let $\widetilde{T} \subset \partial\widetilde{\mathcal{E}}$ be a component of the preimage of T. Let $\widetilde{\mu}$ be a component on $\widetilde{T}$ of the preimage of μ. Using brackets to denote fundamental group elements, we find that $[\widetilde{\mu}]$ is conjugate to $[\mu]^k$ in $\pi(L)$, where $k = \deg q_n|_{p(\widetilde{T})}$, since $[\mu] \in \Gamma'$ by the above. If $\widetilde{\lambda}$ is a component of the preimage of λ, then by the above $[\widetilde{\lambda}]$ is conjugate to $[\lambda]^g$ in $\pi(L)$, where $g = \deg p'|_{q(\widetilde{T})}$.

Let $\widetilde{\mathcal{E}}(\widetilde{\mu})$ be obtained from $\widetilde{\mathcal{E}}$ by Dehn filling along $\widetilde{\mu}$. More precisely, $\widetilde{\mathcal{E}}(\widetilde{\mu})$ is the quotient of $\widetilde{\mathcal{E}} \sqcup \widetilde{V}$, where $\widetilde{V} \cong D^2 \times S^1$, by a homeomorphism $\partial D^2 \times S^1 \to \widetilde{T}$ taking

$\partial D^2 \times \{1\}$ to $\widetilde{\mu}$ and $\{x\} \times S^1$ to $\widetilde{\lambda}$. Then the map $\widetilde{V} \to V$ which is the product of the k-fold branched covering $D^2 \to D^2$ with the g-fold covering $S^1 \to S^1$ extends $\tilde{p}$ to a map $\widetilde{\mathcal{E}}(\widetilde{\mu}) \to \mathcal{E} \cup V$. The extension of $\tilde{p}$ factors on $\widetilde{V}$ as the composition of q_n with a g-fold cover extending p or the composition of p' with a k-fold branched cover extending q.

Extending $\tilde{p}$ across each component of $\partial\widetilde{\mathcal{E}}$ in the manner prescribed above produces a map $\tilde{p}\colon \widetilde{M} \to M$ so that the diagram of the lemma commutes, with p a covering and q a branched covering. Furthermore, it follows from the explicit description above that on each component $\widetilde{V}$ of the preimage of $\mathcal{N}(L)$, $\deg q|_{\widetilde{V}} = \deg q_n|_{p(\widetilde{V})}$. □

Thurston used a version of Proposition 1.2 below to show the reflection orbifold in a right–angled dodecahedron is virtually fibered (cf. [24]); this fact is also used by Boyer-Zhang [3, Cor 1.4]. Our version explicitly describes a fibered cover.

Proposition 1.2. *Suppose $p'\colon M' \to M$ is a g-fold cover and $q_n\colon M_n \to M$ an n-fold branched cover, branched over a link $L \subset M$. If M' fibers over S^1 with fibers transverse to $(p')^{-1}(L)$, then the manifold $\widetilde{M}$ supplied by Lemma 1.1 fibers over S^1 in such a way that $q\colon \widetilde{M} \to M'$ is fiber-preserving.*

Proof. Since $p^{-1}(L)$ is transverse to the fibering of M, for each component V of $\mathcal{N}(L)$, a homeomorphism to $D^2 \times S^1$ may be chosen so that after an ambient isotopy of the fibering, for any component $\widetilde{V}$ of $(p')^{-1}(V)$, each fiber intersects $\widetilde{V}$ in a collection of disjoint disks of the form $D^2 \times \{y\}$ for $y \in S^1$. Then $\mathcal{E}'$ inherits a fibering from M' with the property that each fiber intersects the boundary in a collection of meridians.

By definition, each curve on $\partial\widetilde{\mathcal{E}}$ which bounds a disk in $\widetilde{M}$ is a component of the preimage of a meridian of $\mathcal{E}$. Hence the fibering which $\widetilde{\mathcal{E}}$ inherits from $\mathcal{E}'$ by pulling back using q extends to a fibering of $\widetilde{M}$, which q maps to that of M' by construction. □

We will encounter the following situation: M' is the trivial F-bundle over S^1 for some closed surface F, homeomorphic to $F \times I/((x,1) \sim (x,0))$, and $(p')^{-1}(L)$ consists of simple closed curves in disjoint copies of F. Here $I = [0,1]$. Let $\pi\colon M' \to F$ be projection to the first factor. The second main result of this section describes a property of the collection $\pi((p')^{-1}(L))$ which allows a fibering of M' to be found satisfying the hypotheses of Proposition 1.2.

Proposition 1.3. *Let $M = F \times I/(x,0) \sim (x,1)$, and suppose $L = \{\lambda_1, \ldots \lambda_m\}$ is a link in M such that for each j there exists $t_j \in I$ with $\lambda_j \subset F \times \{t_j\}$, and $t_j \neq t_{j'}$ for $j \neq j'$. Suppose there is a collection of disjoint simple closed curves $\{\gamma_1, \ldots, \gamma_n\}$ on F, each transverse to $\pi(\lambda_j)$ for all j, with the following properties. For each j, there is an i such that $\pi(\lambda_j)$ intersects γ_i, and a choice of orientation of the γ_i and all curves of L may be fixed so that for any i and j, γ_i and $\pi(\lambda_j)$ have equal algebraic and geometric intersection numbers. Then M has a fibering transverse to L.*

The other fiberings needed to prove this theorem may be found by *spinning* annular neighborhoods of the γ_i in the fiber direction. We first saw this technique in [12].

Definition. Let $M = F \times I/(x,0) \sim (x,1)$ be the trivial bundle, and let γ be a simple closed curve in F. Let A be a small annular neighborhood of γ, and fix a marking homeomorphism $\phi\colon S^1 \times I \to A$. We define the fibration obtained by spinning A in the fiber direction to be

$$F_A(t) = ((F - A) \times \{t\}) \bigcup \Phi_t(S^1 \times I),$$

where $\Phi_t(x,s) = (\phi(x,s), \rho(s) + t)$, for t between 0 and 1. Here we take $\rho\colon I \to I$ to be a smooth, nondecreasing function taking 0 to 0 and 1 to 1, which is constant on small neighborhoods of 0 and 1 and has derivative at least 1 on $[1/4, 3/4]$.

Given a collection of disjoint simple closed curves $\gamma_1, \dots, \gamma_n$, one analogously produces a new fibration $F_{A_1,\dots,A_n}(t)$ by spinning an annular neighborhood of each in the fiber direction.

Suppose λ is a simple closed curve in F which has identical geometric and algebraic intersection numbers with the core of each annulus A_i in such a collection; that is, an orientation of λ is chosen so that each oriented intersection with the core of each A_i has positive sign.

Lemma 1.4. *Let λ be such a curve, embedded in M by its inclusion into $F \times \{t_0\}$, $t_0 \in (0,1)$. There is an ambient isotopy which moves λ to be transverse to the fibration $F_{A_1,\dots,A_n}(t)$, and which may be taken to be supported in an arbitrarily small neighborhood of $F \times \{t_0\}$.*

Proof. λ may be isotoped in F so that its intersection with the A_i is of the form $(\{x_1\} \times I) \sqcup \ldots \sqcup (\{x_k\} \times I)$ for some collection $\{x_1, \dots, x_k\}$ of points in their cores. For reference fix a Riemannian metric on F in which the A_i are isometrically embedded with their natural product metric, and choose a smooth unit–speed parametrization $\lambda(t)$ ($t \in I$) so that $\lambda([1/4, 3/4]) = \{x_1\} \times [1/4, 3/4]$. For fixed small $\epsilon > 0$, we embed λ in M with the aid of a map $h_\epsilon\colon I \to I$, defined as follows. Let h'_ϵ be a smooth bump function which takes the value $-\epsilon$ on $[0, 1/4]$ and $[3/4, 1]$, is increasing on $[1/4, 3/8]$ and decreasing on $[5/8, 3/4]$, takes the value 2ϵ on $[3/8, 5/8]$, and has integral equal to 0. Then define h_ϵ by

$$h_\epsilon(s) = t_0 + \int_0^s h'_\epsilon,$$

and let $\lambda_\epsilon(s) = (\lambda(s), h_\epsilon(s))$.

At any point of M, the parametrization of M as $F \times I/(x,1) \sim (x,0)$, gives a natural decomposition of the tangent space. We call *horizontal* the tangent planes to F, and let $\mathbf{t}$ denote the vertical vector pointing upward. In the complement of the vertical tori determined by the $A_i \times I$, the new fiber surface $F_{A_1,\dots,A_n}(t)$ has horizontal tangent planes, for each t. Since the intersection of λ_ϵ with this region is contained in $\lambda_\epsilon([0,1/4] \cup [3/4, 1])$, its tangent vector in this region has $\mathbf{t}$–component $-\epsilon$. Hence intersections in this region between λ_ϵ and copies of the fiber surface $F_{A_1,\dots,A_n}$ are transverse.

For points in A_i, consider the vertical plane spanned by $\mathbf{t}$ and the tangent vector to the I–factor of A_i. Tangent vectors to λ_ϵ at points which lie in $A_i \times I$ lie in this plane with slope between $-\epsilon$ and 2ϵ, possibly greater than $-\epsilon$ only between 1/4 and 3/4. On the other hand, the intersection of the tangent plane to $F_{A_1,\dots,A_n}$ intersects the vertical plane in a line with slope greater than or equal to 0, and greater than or equal to 1 on $[1/4, 3/4]$. Thus as long as $2\epsilon < 1$, any intersection

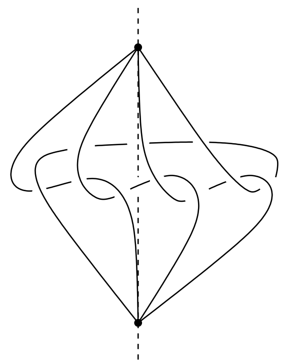

FIGURE 2. Suzuki's Brunnian graph θ_4 and an axis of fourfold symmetry.

of λ_ϵ with a copy of $F_{A_1,\dots,A_n}$ in these regions is transverse as well. The original embedding of λ may clearly be moved to λ_ϵ by a small ambient isotopy, and since λ_ϵ is transverse to each $F_{A_1,\dots,A_n}(t)$, this proves the lemma. □

Proof of Proposition 1.3. Let $\{\gamma_1, \dots, \gamma_n\}$ be a collection satisfying the hypotheses, and let $\{A_i\}$ be a collection of disjoint annular regular neighborhoods of the γ_i in F. By the lemma above, each member λ_j of L may be moved by an ambient isotopy to be transverse to the fibration obtained by spinning each A_i in the fiber direction. Since the components of L lie in disjoint fibers of the original fibration, these isotopies may be taken to have disjoint supports. Then the inverse of their composition, applied to the fibration obtained by spinning A in the fiber direction, produces a new fibration which is transverse to L. □

2. Introducing the tetrus

In this section and the next, we will frequently encounter branched coverings of the form $q\colon M_n \to M$, where M_n and M are manifolds with nonempty boundary. In this case, the branch locus may have components which are properly embedded arcs in M. If T is the branch locus, we require that the regular neighborhood $\mathcal{N}(T)$ have the property that on a component V of $q^{-1}(\mathcal{N}(T))$ projecting to a neighborhood of an arc component of T, there is a homeomorphism $V \to D^2 \times I$ such that q is modeled by the product of the k-fold branched cover $D^2 \to D^2$ with the identity map on I. With the same requirement on circle components of T, and $\mathcal{E}(T)$ defined as before, we remark that Lemma 1.1 holds verbatim in this context.

Thurston constructed a hyperbolic manifold with totally geodesic boundary, which he called the "tripus," from two hyperbolic truncated tetrahedra in Chapter 3 of his notes [25]. A description of the tripus as the complement of a genus two handlebody embedded in S^3 may be found there. In [20], Paoluzzi-Zimmermann generalized this construction, constructing for each $n \geq 3$ and k between 0 and $n-1$ with $(2-k, n) = 1$ a hyperbolic manifold $M_{n,k}$ with geodesic boundary, for which the tripus is $M_{3,1}$. Ushijima extended Thurston's description of the tripus as a handlebody complement in S^3 to show that each $M_{n,1}$, $n \geq 3$, is homeomorphic to the exterior — that is, the complement of a regular neighborhood — of Suzuki's Brunnian graph θ_n [26]. In particular, the tetrus $M_{4,1}$ is the exterior of the graph θ_4 pictured in Figure 2.

There is an order–4 automorphism of the tetrus visible in the figure as a rotation through the dotted axis, which intersects θ_4 only in its two vertices. The quotient of $M_{4,1}$ by the group that this automorphism generates yields a branched cover $q_{4,1}\colon M_{4,1} \to B^3$, branched over the tangle T pictured in Figure 3. In fact Paoluzzi–Zimmermann describe, for each $n \geq 3$ and k with $0 \leq k < n$ and $(2-k, n) = 1$, an n-fold branched cover $q_{n,k}\colon M_{n,k} \to B^3$, branched over T (see [20, Figures 4 & 5]). The quotient map $q_{n,k}$ may be realized by a local isometry to an *orbifold* O_n with geodesic boundary, with underlying topological space B^3 and singular locus T with strings of cone angle $2\pi/n$. We summarize Paoluzzi–Zimmermann's description of the orbifold fundamental group of O_n and its relationship with the fundamental groups of the $M_{n,k}$ in the theorem below; this collects various results in [20].

Theorem 2.1 (Paoluzzi–Zimmermann)**.** *For each $n \geq 3$, the orbifold fundamental group of O_n is presented as*

$$E_n \cong \langle\, X_n, H_n \mid H_n^n = (H_n X_n H_n X_n^{-2})^n = 1 \,\rangle.$$

Each elliptic element of E_n is conjugate to exactly one of H_n or $H_n X_n H_n X_n^{-2}$. For $(2-k, n) = 1$, the fundamental group $G_{n,k} := \pi_1(M_{n,k})$ is the kernel of the projection $\pi_k\colon E_n \twoheadrightarrow \mathbb{Z}_n = \langle h_n \rangle$ given by $\pi_k(X_n) = h_n^k$ and $\pi_k(H_n) = h_n$.

Here we have departed from the notation of Paoluzzi–Zimmermann in distinguishing between presentations of the orbifold fundamental group of O_n for different n, so that Paoluzzi-Zimmermann's "x" is replaced above by our "X_n" and similarly for "h" in their presentation for E_n in [20, p. 120]. Also, we have renamed the generator of $\mathbb{Z}_n$ to h_n.

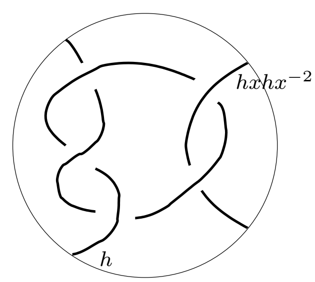

FIGURE 3. A tangle T in the ball B^3, with two arcs labeled by the corresponding elements of $\pi_1(B^3 - T)$

In view of Proposition 1.2, it is important that we have a description of the fundamental group of the exterior of T in B^3. Let $\mathcal{N}(T)$ be a regular neighborhood of T in B^3. Then $\mathcal{N}(T)$ has two components, each homeomorphic to $D^2 \times I$ in such a way that its intersection with T is sent to $\{(0,0)\} \times I$ and its intersection with ∂B^3 to $D^2 \times \{0,1\}$. Then take $\mathcal{E}(T) = \overline{B^3 - \mathcal{N}(T)}$, the exterior of T in B^3, and let $\pi(T) = \pi_1(\mathcal{E}(T))$. We refer by a *meridian of* T to a curve on $\partial\mathcal{N}(T) \cap \mathcal{E}(T)$ of the form $\partial D^2 \times \{y\}$, for $y \in I$. Below we summarize some facts about $\pi(T)$, which may be found for instance in joint work with Eric Chesebro [8, §2].

Lemma 2.1. *The group $\pi(T)$ is free on generators x and h. In $\pi(T)$, the meridians of T are represented by h and $hxhx^{-2}$, and the four-holed sphere $\partial B^3 \cap \mathcal{E}(T)$ is represented by $\Lambda = \langle h, hxhx^{-2}, (xhx)h^{-1}(xhx)^{-1} \rangle$ in $\pi(T)$.*

Using Lemma 2.1, we reinterpret the Theorem of Paoluzzi-Zimmermann below in a way that matches our treatment of branched covers in Section 1.

Lemma 2.2. *For each $n \geq 3$ and k with $(2-k, n) = 1$, let $\Gamma_{n,k}$ be the kernel of the map $\pi_{n,k}\colon\ \pi(T) \twoheadrightarrow \mathbb{Z}_n = \langle h_n \rangle$ given by $x \mapsto h_n^k$, $h \mapsto h_n$, and let $q_{n,k}\colon\ \mathcal{E}_{n,k} \to \mathcal{E}(T)$ be the cover corresponding to $\Gamma_{n,k}$. Then $q_{n,k}$ extends to the branched covering $q_{n,k}\colon\ M_{n,k} \to B^3$ described by Paoluzzi-Zimmermann, where $M_{n,k}$ is obtained from $\mathcal{E}_{n,k}$ by filling each component of the preimage of $\partial\mathcal{N}(T)\cap\mathcal{E}(T)$ with a copy of $D^2\times I$.*

Proof. The homomorphism $\pi_{n,k}$ described in the lemma takes the elements h and $hxhx^{-2}$ to h_n and $(h_n)^{2-k}$, respectively, each of which generates $\mathbb{Z}_n$ when $2-k$ is relatively prime to n. Since the meridians of $\mathcal{E}(T)$ are represented by h and $hxhx^{-2}$, it follows that each meridian has connected preimage in $\mathcal{E}_{n,k}$, represented in $\Gamma_{n,k}$ by h^n and $(hxhx^{-2})^n$, respectively.

If U is a component of $\partial\mathcal{N}(T)\cap\mathcal{E}(T)$, then it is homeomorphic to $\partial D^2 \times I$, and by the paragraph above $(q_{n,k})^{-1}(U)$ is a connected n-fold cover of U, modeled by the product of the n-fold cover $\partial D^2 \to \partial D^2$ with the identity map $I \to I$. Thus after filling each component of $\partial\mathcal{N}(T)\cap\mathcal{E}(T)$ and its preimage under $q_{n,k}$ with a cylinder $D^2 \times I$, $q_{n,k}$ extends to a branched cover modeled on the cylinders by the product of the standard n-fold branched cover $D^2 \to D^2$ with the identity map $I \to I$.

By our descriptions of $\mathcal{N}(T)$ and $\mathcal{E}(T)$, the image of this branched cover is homeomorphic to B^3, with branching locus T. We claim that the domain is homeomorphic to $M_{n,k}$. There is a map from $\pi(T)$ onto the orbifold group E_n given by sending x and h to X_n and H_n, respectively. Then $\pi_{n,k}$ factors as this projection followed by the map π_k defined by Paoluzzi-Zimmermann. The description of E_n thus implies that $G_{n,k}$ is the quotient of $\Gamma_{n,k}$ by the normal closure of $\{h^n, (hxhx^{-2})^n\}$, and the claim follows. □

The *double* branched cover $M_2 \to B^3$, branched over T, was not addressed in [20] since it does not admit the structure of a hyperbolic manifold with totally geodesic boundary. In fact, it is homeomorphic to the trefoil knot exterior, and we will describe its Seifert fibered structure and the preimage of T in detail in Section 3. Below we give a description consistent with that of Lemma 2.2.

Lemma 2.3. *Let Γ_2 be the kernel of the map $\pi_2\colon\ \pi(T) \twoheadrightarrow \mathbb{Z}_2 = \{0,1\}$ given by $x \mapsto 1$, $h \mapsto 1$, and let $q_2\colon\ \mathcal{E}_2 \to \mathcal{E}(T)$ be the cover corresponding to Γ_2. Then q_2 extends to the unique twofold branched cover $q_2\colon\ M_2 \to B^3$, after filling components of the preimage of $\partial\mathcal{N}(T)\cap\mathcal{E}(T)$ with copies of $D^2 \times I$.*

Remark. The uniqueness property above is a standard feature of double branched covers. We find it useful in the next section, where we describe M_2 by other means.

Proof. Let $q\colon\ M \to B^3$ be a twofold branched cover with branch locus T. The associated cover $q\colon\ \mathcal{E} \to \mathcal{E}(T)$ corresponds to a subgroup $\Gamma < \pi(T)$ which is of index 2 and hence normal. Each element of $\pi(T)$ representing a meridian of T must map nontrivially under the quotient $\pi(T) \to \pi(T)/\Gamma_2 \simeq \mathbb{Z}_2$, since q branches nontrivially over each component of T. By the description in Lemma 2.1, it follows that h and h^2x^{-1} map nontrivially, hence that each of h and x map to the generator. Thus the only twofold branched cover of B^3, branched over T, is $q_2\colon\ M_2 \to B^3$ as described in the lemma. □

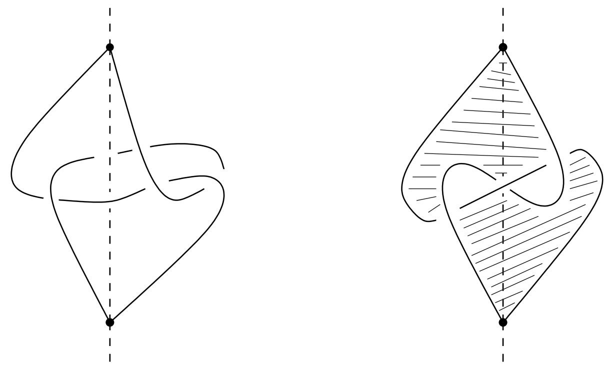

FIGURE 4. Two views of θ_2, and a Seifert surface for the trefoil exterior.

The corollary below is the reason is the reason that evenfold branched covers are particularly amenable to the techniques of this paper.

Corollary 2.2. *For each* $n \geq 2$ *and* k *with* $(2-k, 2n) = 1$*, the map* π_2 *of Lemma 2.3 factors as* $\rho_n \circ \pi_{2n,k}$*, where* $\pi_{2n,k}$ *is as described in Lemma 2.2 and* $\rho_n \colon \mathbb{Z}_{2n} \twoheadrightarrow \mathbb{Z}_2$*. Hence* $\Gamma_{2n,k} < \Gamma_2$*, and the resulting covering map* $\mathcal{E}_{2n,k} \to \mathcal{E}_2$ *extends to a branched covering* $M_{2n,k} \to M_2$ *after filling components of the preimage of* $\partial\mathcal{N}(T) \cap \mathcal{E}(T)$ *with copies of* $D^2 \times I$*.*

3. Virtually fibering the doubled tetrus

In this section we construct a fibered cover for $DM_{4,1}$ using the methods of Section 1. We motivate the strategy by an appeal to Figure 4, by which one may easily identify M_2. The quotient of the pair (S^3, θ_4) by 180-degree rotation in the dotted axis of Figure 2 is (S^3, θ_2), pictured on the left-hand side of Figure 4. A small isotopy of θ_2 yields the graph on the right-hand side of Figure 4, whose exterior is clearly that of the trefoil knot.

The dotted axis is the fixed locus of the unique strong involution of the trefoil, induced by a further 180-degree rotation, with quotient a single arc in S^3. The exterior of this arc is thus homeomorphic to B^3, and an ambient isotopy taking this arc to a "standard" position moves the dotted axis to the tangle T of Figure 3. (For a depiction of this we refer the reader to [18, Figure 3(c)].) It follows from uniqueness of the double branched cover that the sequence of induced maps $\mathcal{E}(\theta_4) \to \mathcal{E}(\theta_2) \to B^3$ is the sequence $M_{4,1} \to M_2 \to B^3$ described in Corollary 2.2.

The trefoil knot exterior is well known to fiber over the circle, with fiber the shaded one-holed torus of Figure 4, and monodromy map of order 6 (see eg. [21, Ch. 10.I]). The branch locus for q_2, the intersection of the dotted axis of the figure with the exterior of θ_2, has one component visibly contained in the pictured fiber surface. In fact, the other component is isotopic into a different fiber. This follows from the fact that a fibered knot exterior has a unique fibering up to isotopy (see [7, Proposition 5.10] and the remarks below it), so that after an isotopy the strong involution takes fibers to fibers. Hence there is a sixfold cover $p' \colon M' \to M_2$ which is trivially fibered over the circle in such a way that the preimage of the dotted axis is contained in a disjoint union of fibers.

Our strategy is thus to apply Lemma 1.1 to find a map $\tilde{p}\colon \widetilde{M} \to M$ which completes the diamond of maps $M_{4,1} \to M_2$ and $M' \to M_2$ described above. This is the subject of Lemma 3.3. Doubling across boundaries yields a corresponding diamond, such that the double DM' of M' is trivially fibered over the circle and the preimage of the branch locus for $DM_{4,1} \to DM_2$ consists of simple closed curves in disjoint fibers. The proof of Theorem 0.1 then reduces to an application of Propositions 1.3 and 1.2.

Our main goal at the beginning of this section is to give a description of M_2 and its fibering in such a way that Proposition 1.3 can be easily applied to DM'. To this end, Lemma 3.2 and the material leading up to it describe the fiber surface F for M_2 as a horizontal surface in a Seifert fibering of M_2 over a disk with two exceptional fibers. Our construction is an ad hoc version of one due to Montesinos [17], which describes a Seifert fibered structure on the double branched cover of a link built as a sum of *rational tangles*, introduced by Conway [9]. T is represented as $3\,0 + 2\,0$ in Conway's notation (see [9, Fig. 1,2,3]), where $3\,0$ and $2\,0$ associate to the rational numbers $1/3$ and $1/2$, respectively.

Let $V = D^2 \times S^1$ be the solid torus, embedded in $\mathbb{C}^2$ as the cartesian product of the unit disk in $\mathbb{C}$ with its boundary, and oriented as a product of the standard orientation on D^2 with the boundary orientation S^1 inherits as ∂D^2. Define the *complex conjugation-induced* involution of V by $(z, w) \mapsto (\bar{z}, \bar{w})$. The fixed set is $S = \{(r, \pm 1) \mid r \in [-1, 1]\}$, a disjoint union of two arcs properly embedded in V. The quotient map $q\colon V \to V/((z, w) \sim (\bar{z}, \bar{w})) \cong B^3$ is a twofold branched covering with branching locus S. Each meridian disk $D^2 \times \{\pm 1\}$ of V is mapped by q to a disk in B^3 which determines an isotopy rel endpoints between an arc of $q(S)$ and an arc on ∂B^3. That is, $q(S)$ is the trivial two–string tangle B^3.

A rational number p/q in lowest terms determines a Seifert fibering of V, with an *exceptional fiber* parametrized by $\gamma_0(t) = (0, e^{2\pi i t})$, $t \in I$, and *regular fibers* parametrized by

$$\gamma_z(t) = \left(z\, e^{2\pi i \cdot pt}, e^{2\pi i \cdot qt}\right), \quad t \in I, \tag{1}$$

for $z \in D^2 - \{0\}$. Then z and w in D^2 determine the same fiber if and only if $w = ze^{2\pi i k/q}$ for some $k \in \mathbb{Z}$. Hence for any $w \in S^1$, the quotient map sending each fiber to a point restricts on $D^2 \times \{w\}$ to a q-fold branched covering, branched at the origin. We let $V_{p/q}$ denote V equipped with this fibering, and divide $\partial V_{p/q}$ into annuli $A_{p/q}$ and $B_{p/q}$, parametrized as follows. Define a model annulus $A = I \times I/(x, 0) \sim (x, 1)$, inheriting a "vertical" fibering by circles from arcs $\{x\} \times I$, a "horizontal" fibering from arcs of the form $I \times \{y\}$, and an orientation from the standard orientation on $I \times I$. Define $\phi_{p/q}\colon A \to \partial V_{p/q}$ by

$$\phi_{p/q}(x, y) = \left(e^{2\pi i\left(\frac{1-2x}{4q}\right)} e^{2\pi i \cdot py}, e^{2\pi i \cdot qy}\right).$$

Choose a and b such that $ap + bq = 1$, and define $\psi_{p/q}\colon A \to \partial V_{p/q}$ by

$$\psi_{p/q}(x, y) = \left(e^{2\pi i\left(\frac{2x+1}{4q}\right)} e^{2\pi i \cdot p\left(y - \frac{a}{2q}\right)}, e^{2\pi i \cdot q\left(y - \frac{a}{2q}\right)}\right).$$

Then $\phi_{p/q}$ and $\psi_{p/q}$ have the following properties.

(1) Taking $A_{p/q} \doteq \phi_{p/q}(A)$ and $B_{p/q} \doteq \psi_{p/q}(A)$, we have $A_{p/q} \cup B_{p/q} = \partial V_{p/q}$, and $A_{p/q} \cap B_{p/q} = \phi_{p/q}(\partial A) = \psi_{p/q}(\partial A)$.

(2) Each of $\phi_{p/q}$ and $\psi_{p/q}$ takes a vertical fiber of A to a Seifert fiber of $V_{p/q}$ and a horizontal fiber to a closed arc in $\partial D^2 \times \{y\}$ for some $y \in S^1$.

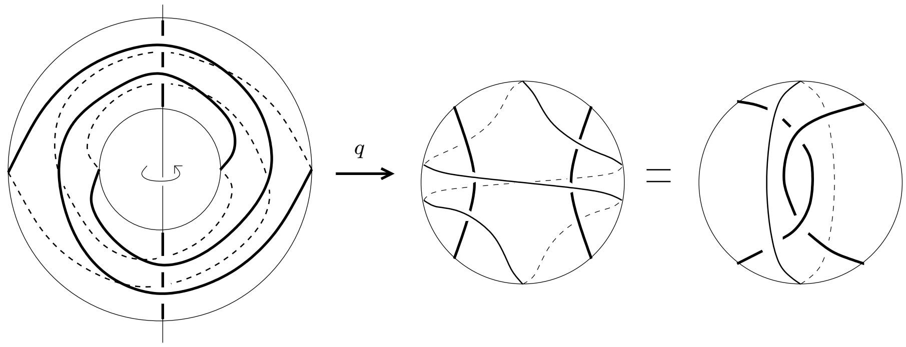

FIGURE 5. The double branched cover of the rational tangle 1/2.

(3) Giving $A_{p/q}$ and $B_{p/q}$ the boundary orientations from $V_{p/q}$, $\phi_{p/q}$ reverses and $\psi_{p/q}$ preserves orientation.
(4) The complex conjugation-induced involution on $V_{p/q}$ commutes with the map $(x,y) \mapsto (1-x, 1-y)$ under each of $\phi_{p/q}$ and $\psi_{p/q}$. In particular, each of $A_{p/q}$ and $B_{p/q}$ contains two points of $S \cap \partial V_{p/q}$, and $(1,1) \in A_{p/q}$.

The map q determines a double branched cover of $V_{1/2}$ to the ball, branched over the rational tangle 1/2, as illustrated in Figure 5. In the figure, the two parallel simple closed curves comprising $A_{1/2} \cap B_{1/2}$ are drawn on $\partial V_{1/2}$, projecting to the boundary of the indicated disk on B^3. A similar picture holds for the double branched cover of $V_{1/3}$ to the 1/3 rational tangle. An appeal to Property 4 of the parametrizations above thus yields the following lemma.

Lemma 3.1. *Define $M_2 = V_{1/3} \cup_{\phi_2\psi_3^{-1}} V_{1/2}$. There is a branched cover $q_2\colon M_2 \to B^3$, branched over T, which restricts on each V_i to q.*

Property 2 of the parametrizations ψ_3 and ϕ_2 implies that M_2 inherits the structure of a Seifert fibered space from $V_{1/3}$ and $V_{1/2}$. The lemma below describes a foliation of M_2 by surfaces, each meeting each Seifert fiber transversely.

Lemma 3.2. *Let $H_m = D^2 \times \{e^{2\pi i \frac{m-1}{2}}\} \subset V_{1/3}$, $m = 0,1$, and $S_n = D^2 \times \{e^{2\pi i \frac{n}{3}}\} \subset V_{1/2}$, $n = 0,1,2$. Then $F = (\bigcup H_m) \cup (\bigcup S_n)$ is a connected surface homeomorphic to a one-holed torus, which is a fiber in the fibration of M_2 that restricts on $V_{1/3}$ or $V_{1/2}$ to the foliation by disks $D^2 \times \{y\}$. A map $\sigma\colon F \to F$ is determined by the following combinatorial data: $\sigma(H_m) = H_{1-m}$ for $m = 0,1$, and $\sigma(D_n) = D_{n+1}$ for $n = 0,1,2$ (take $n+1$ modulo three), so that $M \cong F \times [0,1]/((x,0) \sim (\sigma(x),1))$.*

Proof. By Property 2 of $\phi_{1/2}$ and $\psi_{1/3}$, for each $y \in S^1$, the gluing map $\phi_{1/2}\psi_{1/3}^{-1}$ takes components of $\partial D^2 \times \{y\} \cap B_{1/3} \subset V_{1/3}$ to components of $\partial D^2 \times \{y'\} \cap A_{1/2} \subset V_{1/2}$, for y' determined by y. It follows that the foliations of $V_{1/3}$ and $V_{1/2}$ by disks of the form $D^2 \times \{y\}$ join in M_2 to yield a foliation by surfaces. We take F to be the surface in this foliation containing $D^2 \times \{1\}$ in $V_{1/3}$, and illustrate its combinatorics in Figure 6.

We depict the disks H_m as hexagons, since each intersects each of $A_{1/3}$ and $B_{1/3}$ in three arcs of its boundary. Applying $\psi_{1/3}^{-1}$ to a component of $H_m \cap B_{1/3}$ yields

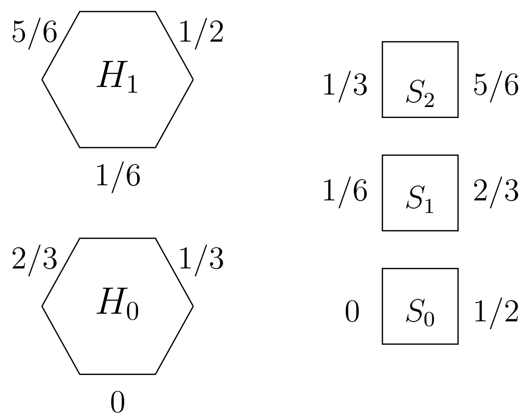

FIGURE 6. The surface F in M_2.

an arc of the form $I \times \{h\}$ for some $h \in I$; in the figure, we have labeled each arc of $H_m \cap B_{1/3}$ by the corresponding h. Each "square" S_n intersects $A_{1/2}$ in two arcs of its boundary, labeled in Figure 6 by the height of their images under $\phi_{1/2}^{-1}$. We assign each H_m or S_n the standard orientation from D^2, and picture them in Figure 6 with the orientation inherited from the page. Then each labeled edge of H_m is identified to that of S_n with the same label, in orientation–reversing fashion, by $\phi_{1/2}\psi_{1/3}^{-1}$. Their union F is now easily identified as a one–holed torus.

Since the Seifert fibers of $V_{1/3}$ and $V_{1/2}$ are transverse to the disks $D^2 \times \{y\}$, Seifert fibers of M_2 transversely intersect each surface in the foliation described above. There is a quotient map π, taking M_2 to a closed one-manifold (that is, S^1), determined by crushing each surface in the foliation described above to a point. Then F is the preimage under π of a point in S^1, so cutting M_2 along F yields a surface bundle over I, necessarily of the form $F \times I$. With F oriented as prescribed above, the normal orientation to F in M_2 is the upward direction along Seifert fibers parameterized as in (1). The "first return map" $\sigma\colon F \to F$ is prescribed as follows: from $x \in F$, move in the normal direction along the Seifert fiber through x until it again intersects F; the point of intersection is $\sigma(x)$. It describes a monodromy for the description of M_2 as a bundle over S^1.

From the description of σ and the fiber-preserving parameterizations $\psi_{1/3}$ and $\phi_{1/2}$, we find that for a point x on an arc labeled by h in Figure 6, $\sigma(x)$ is the corresponding point on the arc labeled by $h + 1/6$ (modulo 1). This models the behavior of $\sigma(x)$ on the regular fibers. The intersection of F with the singular fiber in $V_{1/3}$ is the disjoint union of the centers of H_0 and H_1, which are thus interchanged by σ. An analogous description holds for the intersection of F with the singular fiber in $V_{1/2}$, yielding the description of σ in the statement of the lemma. Now since $V_{1/3}$ is cut by its intersection with F into two cylinders, and $V_{1/2}$ is cut into three, and these are identified along vertical annuli in their boundaries by $\phi_{1/2}\psi_{1/3}^{-1}$ to form M_2 cut along F, the description of M_2 as a fiber bundle with fiber F follows.

By construction, the components of $q_2^{-1}(T)$ in M_2 intersect each of $V_{1/3}$ and $V_{1/2}$ in the fixed set of the complex conjugation-induced involution. In $V_{1/3}$, the component $[-1,1]\times\{-1\}$ is contained in H_0, and $[-1,1]\times\{1\}\subset H_1$. In $V_{1/2}$, the component $[-1,1]\times\{1\}$ is contained in S_0, and $\phi_{1/2}\psi_{1/3}^{-1}$ takes its endpoints to endpoints of the components in $V_{1/3}$. □

Note that $q_2^{-1}(T)$ intersects each of $V_{1/3}$ and $V_{1/2}$ in the fixed locus of the complex conjugation-induced involution. Since the components of this locus lie in the disks $D^2\times\{\pm1\}$, it follows that $q_2^{-1}(T)$ lies in fibers of the fibration $M_2\to S^1$ described in Lemma 3.2. The description of Lemma 3.2 also implies that σ is periodic with order 6. Therefore M_2 has a sixfold cover $p'\colon M'\to M_2$ which is trivially fibered; that is, $M'\cong F\times S^1$, such that components of $(q_2\circ p')^{-1}(T)$ lie in disjoint fibers.

The uniqueness property of Lemma 2.3 implies that the manifolds M_2 described there and in Lemma 3.1 are homeomorphic as branched covers. In particular, the map $q_{4,1}\colon M_{4,1}\to B^3$ described in Lemma 2.2 factors through q_2 described in Lemma 3.1. Let $q_{2,1}\colon M_{4,1}\to M_2$ be the map whose existence follows from Corollary 2.2, such that $q_{4,1}=q_2\circ q_{2,1}$. Then $q_{2,1}$ is a twofold branched cover, branched over $(q_2)^{-1}(T)$.

Recall, from the first paragraph of Section 2, that Lemma 1.1 applies in the context of manifolds with nonempty boundary.

Lemma 3.3. *Let $\tilde{p}\colon \widetilde{M}\to M_2$ be the map produced by Lemma 1.1, such that $\tilde{p}=p'\circ q=q_{2,1}\circ p$ for a branched cover $q\colon \widetilde{M}\to M'$ and a cover $p\colon \widetilde{M}\to M_{4,1}$:*

$$\begin{tikzcd} \widetilde{M} \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, "\tilde{p}"] & M' \arrow[d, "p'"] & \\ M_{4,1} \arrow[r, "q_{2,1}"] & M_2 \arrow[r, "q_2"] & B^3 \end{tikzcd}$$

Then p has degree 6 and $\partial\widetilde{M}$ is connected, with genus 13.

Remark. For $n\geq 2$, replacing $M_{4,1}$ by $M_{2n,1}$ above yields a cover $p\colon \widetilde{M}\to M_{2n,1}$, again of degree 6, such that $\partial\widetilde{M}$ is connected and has genus $12n-11$.

Proof. Most of the content of this lemma follows immediately from Lemma 1.1. The assertions that p has degree 6 and $\widetilde{M}$ has connected boundary use two additional observations:

(1) Each component of $(q_2)^{-1}(\mathcal{N}(T))$ has connected preimage in $M_{4,1}$, mapping to it with degree 2.
(2) M' has connected boundary, and each component of $(q_2)^{-1}(T)$ has six preimage components in M', each mapped homeomorphically by p'.

That item (1) above is true follows from the more general fact, recorded in Lemma 2.2, that for any n, each component of T has connected preimage in $M_{n,k}$ under $q_{n,k}$. Item (2) obtains from the fact that each component of $(q^2)^{-1}(T)$ is contained in a copy of the fiber surface F, a one-holed torus, and $M'\cong F\times S^1$.

Thus let $\widetilde{V}$ be a component of $\tilde{p}^{-1}(q_2^{-1}(T))$. By the final assertion of Lemma 1.1, q maps $\widetilde{V}$ to its image in M' with the same degree as that of $q_{2,1}|_{p(\widetilde{V})}$. This is 2 by observation (1) above; hence q has degree 2. (The degree is at most two, since the degree of $\tilde{p}$ is at most 12, and p' has degree 6.) It follows, furthermore, that q maps any component of $\partial\widetilde{M}$ which intersects $\widetilde{V}$ onto its image with degree 2. Thus since $\partial M'$ is connected and intersects $q(\widetilde{V})$, $\partial\widetilde{M}$ is connected as well.

Since $q_{2,1}$ has degree 2 and $\widetilde{p}$ has degree 12, p has degree 6. The genus of $\partial\widetilde{M}$ may be computed using the fact that it is a connected cover, with degree 6, of the genus 3 surface $\partial M_{4,1}$. $\square$

We now turn from consideration of the manifolds-with-boundary $M_{n,k}$ to their doubles, as defined at the beginning of the paper. It is clear from the definition that a map $f\colon M \to N$ between manifolds with boundary determines a map, which we again denote $f\colon DM \to DN$, between doubles, and that the map between doubles inherits the property of being a cover or branched cover of degree n from the original map.

Proposition 3.4. *Taking the branched cover $q_{2,1}\colon DM_{4,1} \to DM_2$ and the cover $p'\colon DM' \to DM_2$ to be determined by the corresponding maps on $M_{4,1}$ and M', respectively, the map supplied by Lemma 1.1 is $\tilde{p}\colon D\widetilde{M} \to DM_2$.*

The point of this proposition is that Lemma 1.1 is "doubling equivariant"; that is, applying it and then doubling the resulting diagram of (branched) covers yields the same result as doubling first and then applying it. It is a consequence of the normal form theorem for free products with amalgamation.

Fact. Let A and B be groups sharing a subgroup C, and let $A *_C B$ be the free product of A with B, amalgamated over C. If $\pi\colon A *_C B \to K$ is an epimorphism to a finite group K such that $\pi(C) = K$, then

$$\ker\pi = \langle \ker\pi|_A, \ker\pi|_B\rangle \simeq (\ker\pi|_A) *_{\ker\pi|_C} (\ker\pi|_B).$$

Proof of Fact. It is clear that $\langle \ker\pi|_A, \ker\pi|_B\rangle$ is contained in $\ker\pi$. We claim equality; this follows from the normal form theorem for free products with amalgamation (see [14, Ch. IV, §2]). Fix sets S_A and S_B of right coset representatives for C in A and B, respectively. Then the normal form theorem asserts that each $g \in A *_C B - \{1\}$ has a *normal form*, which is a unique expression $g = cs_1s_2\cdots s_n$ for some $n \geq 1$, where $c \in C$ and each s_i is in S_A or S_B, with $s_i \in S_A$ if and only if $s_{i+1} \in S_B$. We will call n the *length* of g, and note that the claim is immediate if g has length 1.

If $g \in \ker\pi$ has length $n > 1$, then write g in normal form as above, and let $c_0 \in C$ have the property that $\pi(c_0) = \pi(cs_1\cdots s_{n-1})$. Taking $g_0 = cs_1\cdots s_{n-1}c_0^{-1}$ and $g_1 = c_0g_n$, we may write $g = g_0g_1$ as a product of words in $\ker\pi$. It is evident that g_1 has length 1 and easily proved, by passing elements of C to the left, that g_0 has length at most $n-1$. Then by induction, each of g_0 and g_1 is in $\langle \ker\pi|_A, \ker\pi|_B\rangle$, so g is as well.

The normal form theorem also implies that the naturally embedded subgroups A and B in $A *_C B$ intersect in C. Therefore $\ker\pi|_A \cap \ker\pi|_B = \ker\pi|_C$, and it follows again from the normal form theorem that the inclusion-induced map $(\ker\pi|_A) *_{\ker\pi|_C} (\ker\pi|_B) \to \langle \ker\pi|_A, \ker\pi|_B\rangle$ is an isomorphism. $\square$

Proof of Proposition 3.4. Recall that we have identified the exterior of T, $\mathcal{E}(T)$, with $\overline{B^3 - \mathcal{N}(T)}$, where $\mathcal{N}(T)$ is a regular neighborhood of T in B^3, with two components homeomorphic to $D^2 \times I$. The double of B^3 is homeomorphic to S^3, T doubles yielding the link L of Figure 1, and $\mathcal{N}(T)$ doubles yielding a regular neighborhood $\mathcal{N}(L)$. Each component of $\mathcal{N}(L)$ is homeomorphic to $D^2 \times S^1$, with the corresponding component of $\mathcal{N}(T)$, homeomorphic to $D^2 \times I$, mapping in by $(x, y) \mapsto (x, e^{\pi i y})$ and its mirror image by $(x, y) \mapsto (x, e^{-\pi i y})$. Using this description of $\mathcal{N}(L)$, the link exterior $\mathcal{E}(L)$ is the double of $\mathcal{E}(T)$ across the subsurface $\partial B^3 \cap \mathcal{E}(T)$ of $\partial\mathcal{E}(T)$, and meridians of T in $\mathcal{E}(T)$ are meridians of L in $\mathcal{E}(L)$.

Using the above description, van Kampen's theorem describes $\pi(L)$ as a free product with amalgamation, $\pi(L) \simeq \pi(T) *_\Lambda \overline{\pi(T)}$, across the subgroup Λ from Lemma 2.1, which corresponds to $\partial B^3 \cap \mathcal{E}(T)$. Here $\overline{\pi(T)} = \{\bar{g} \,|\, g \in \pi(T)\}$ is $\pi_1(\overline{\mathcal{E}(T)})$, isomorphic to π_T. There is a "doubling involution" r on $\pi(L)$ determined by $r(g) = \bar{g}$, $g \in \pi(T)$. This is induced by the doubling involution exchanging $\mathcal{E}(T)$ with $\overline{\mathcal{E}(T)}$ in $\mathcal{E}(L)$, fixing their intersection $\partial B^3 \cap \partial\mathcal{E}(T)$.

The projections $\pi_{n,k}\colon \pi(T) \to \mathbb{Z}_n$ and $\pi_2\colon \pi(T) \to \mathbb{Z}_2$, respectively, defined in Lemma 2.2 and Lemma 2.3 respectively, uniquely determine corresponding projections on $\pi(L)$ with the property that $\pi_{n,k} \circ r = \pi_{n,k}$ and that respectively $\pi_2 \circ r = \pi_2$. Since Λ contains the meridian representatives h and $hxhx^{-2}$, each such projection maps it onto the image of $\pi(L)$. Then by the fact above, we have

$$D\Gamma_{n,k} \doteq \ker \pi_{n,k} = \langle \Gamma_{n,k}, \overline{\Gamma}_{n,k} \rangle \simeq \Gamma_{n,k} *_{\Lambda_{n,k}} \overline{\Gamma}_{n,k},$$

where $\Lambda_{n,k} \doteq \ker \pi_{n,k}|_\Lambda$. (The corresponding fact holds for $D\Gamma_2 \doteq \ker \pi_2$.)

We now define $D\mathcal{E}_{n,k}$ (respectively, $D\mathcal{E}_2$) to be the double of $\mathcal{E}_{n,k}$ (resp. $\mathcal{E}_2$) across the subsurface $\partial\mathcal{E}_{n,k} \cap \partial M_{n,k} \subset \partial\mathcal{E}_{n,k}$ (resp. $\partial\mathcal{E}_2 \cap \partial M_2$). This notation is somewhat abusive, since this subsurface does not occupy all of $\partial\mathcal{E}_{n,k}$, but we note that it is represented in $\Gamma_{n,k}$ by $\Lambda_{n,k}$, since its complement is $(q_{n,k})^{-1}(\partial\mathcal{N}(T) \cap \mathcal{E}(T))$. Then $D\mathcal{E}_{n,k}$ (respectively, $D\mathcal{E}_2$) covers $\mathcal{E}(L)$, and the description above makes clear that this is the cover corresponding to $D\Gamma_{n,k}$ (resp. $D\Gamma_2$). Hence filling $D\mathcal{E}_{n,k}$ (resp. $D\mathcal{E}_2$) along preimages of meridians of L yields the branched cover $DM_{n,k}$ (resp. DM_2) defined in the statement of the proposition.

We make a similar claim regarding $p'\colon DM' \to DM_2$. The cover $p'\colon \mathcal{E}' \to \mathcal{E}_2$ is regular, corresponding to the kernel of a map $\pi'\colon \Gamma_2 \to \mathbb{Z}_6$, and since $\partial\mathcal{E}' \cap \partial M'$ is connected, it corresponds to a subgroup $\Lambda' = \ker \pi'|_{\Lambda_2}$ of index 6 in Λ_2. Then defining $\pi'\colon \Gamma_2 *_{\Lambda_2} \overline{\Gamma}_2 \to \mathbb{Z}_6$ by requiring $\pi' \circ r = \pi'$, we argue as above to show that $p'\colon DM' \to DM_2$ is obtained by filling the cover of $D\mathcal{E}_2$ corresponding to $\ker \pi'$ along preimages of meridians.

Lemma 3.3 implies that $\widetilde{\Gamma} = \Gamma_{4,1} \cap \Gamma'$ has index 2 in Γ', and that $\widetilde{\Lambda} = \Gamma_{4,1} \cap \Lambda'$ has index 2 in Λ'. Then it follows as above that the subgroup $D\widetilde{\Gamma} \doteq D\Gamma_{4,1} \cap D\Gamma' = \langle \widetilde{\Gamma}, r(\widetilde{\Gamma}) \rangle$, and that it is isomorphic to the free product of $\widetilde{\Gamma}$ with itself, amalgamated across $\widetilde{\Lambda}$. The proposition follows. $\square$

Proposition 3.4 supplies a diamond of maps to which Proposition 1.2 may be applied. We thus prove Theorem 0.1 below by using Proposition 1.3 to find a fibering of M' transverse to the preimage of L.

Proof of Theorem 0.1. By Lemma 3.2, M_2 is homeomorphic to a bundle over S^1 with fiber the surface F depicted in Figure 6 and monodromy $\sigma\colon F \to F$. The fibers of M_2 join to yield a fibering of DM' with fiber surface DF, the double of F, and monodromy map which we will call $D\sigma$. This is pictured in Figure 7. Here the hexagons and squares to the left of the vertical line are fitted together along labeled arcs as in Figure 6 forming a copy of F, and the hexagons and squares to the right of the vertical line are fitted together along their labeled edges forming a copy of $\overline{F}$. To form DF, each unlabeled edge is identified with its correspondent by reflection through the vertical line. $D\sigma$ is the map that restricts on F to σ and is equivariant with respect to the doubling involution, hence is itself of order 6.

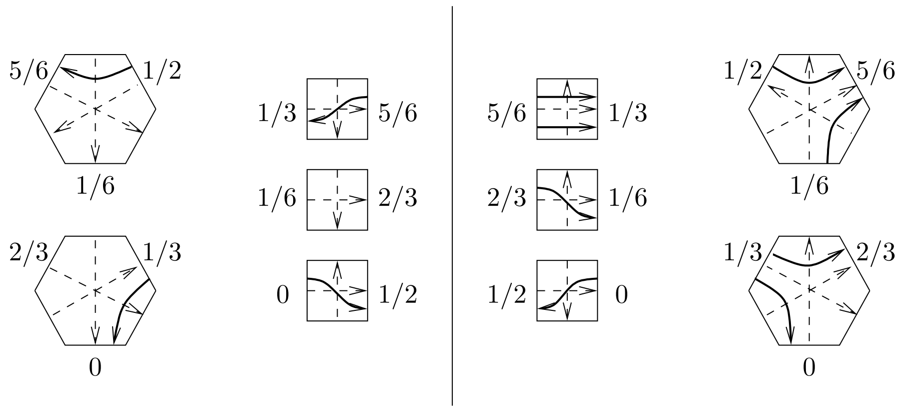

Figure 7. The collection of transverse curves in DF

Recall that $p'\colon M' \to M_2$, defined below Lemma 3.2, is the sixfold cover of M_2 which is trivially fibered over S^1 with fiber F. Since DM' is the double of M', it is trivially fibered over S^1 with fiber DF. Using Lemma 3.2, we may identify DM_2 with $DF \times I/((x,0) \sim (D\sigma(x),1))$. Then identifying DM' with $DF \times I/((x,0) \sim (x,1))$, the covering map is given by $(x,y) \mapsto ((D\sigma)^k(x), 6(y - \frac{k}{6}))$ for $y \in [\frac{k}{6}, \frac{k+1}{6}]$, $0 \le k < 6$. Since the components of $(q_2)^{-1}(T)$ lie in disjoint copies of the fiber surface F for M_2, the components of $q_2^{-1}(L)$ lie in disjoint copies of $DF \subset M$. Take $\pi\colon DM' \to DF$ to be projection onto the first factor. Then by the description above, if λ is a component of $(q_2 \circ p')^{-1}(L) \subset M'$, $\pi(\lambda)$ is a simple closed curve on DF and $D\sigma$ takes $\pi(\lambda)$ to $\pi(\lambda')$, where λ' is another component of $(q_2 \circ p')^{-1}(L)$.

To understand $(q_2 \circ p')^{-1}(L)$, we first describe $(q_2)^{-1}(T)$ in M_2. This the fixed set of the involution which restricts on each of $V_{1/3}$ and $V_{1/2}$ to the complex conjugation-induced involution. The fixed arc $[-1,1] \times \{1\} \subset V_{1/3}$ lies in the disk H_0, running from the midpoint of the side labeled 0 to the midpoint of the opposite, unlabeled side in Figure 6. The other arc $[-1,1] \times \{-1\} \subset V_{1/3}$ runs from the midpoint of the side of H_1 labeled 1/2 to the midpoint of the opposite side. The arc $[-1,1] \times \{1\} \subset V_{1/2}$ lies in S_0, joining the midpoints of the sides labeled 0 and 1/2. Thus the union of these three arcs is an arc properly embedded in M_2, comprising one component of the preimage of T.

The other component of $q_2^{-1}(T)$ is the arc $[-1,1] \times \{-1\} \subset V_{1/2}$, with endpoints in $B_{1/2} \subset \partial M_2$. This lies in the disk $D^2 \times \{-1\}$ in $V_{1/2}$, midway between S_1 and $\sigma(S_1) = S_2$. Thus using the description from Lemma 3.2 of M_2 as $F \times I/(x,0) \sim (\sigma(x),1)$, the second arc of $q_2^{-1}(T)$ lies in $S_1 \times \{1/2\}$, joining midpoints of the unlabeled boundary components.

Now in DM_2, $(q_2)^{-1}(L)$ is the double of $(q_2)^{-1}(T)$, with one component in $DF \times \{0\}$ and one in $DF \times \{1/2\}$. Then $(q_2 \circ p')^{-1}(L) \subset DM'$ has twelve components in disjoint copies of DF. This set is depicted by the dashed arcs in Figure 7. If α is such an arc, a simple closed curve on DF containing α may be obtained by taking the union of a collection of arcs $\mathcal{A}$, of minimal cardinality such that $\alpha \in \mathcal{A}$ and for each $\beta \in \mathcal{A}$, the arcs meeting β at its endpoints are also in $\mathcal{A}$. Inspection reveals six such simple closed curves, permuted by $D\sigma$, each of the form $\pi(\lambda)$ for some component λ of $(q_2 \circ p')^{-1}(L)$. There are six, rather than twelve, because $(D\sigma)^3$ is the hyperelliptic involution of DF, which preserves simple closed curves.

The bold arcs depicted in Figure 7 join to produce two disjoint simple closed curves in DF transverse to this collection. We have indicated orientations with arrows so that, giving DF the standard orientation from the page, the algebraic and geometric intersection numbers of each bold curve with each dashed curve coincide. Thus by Proposition 1.3, spinning annuli along the bold curves yields a fibration of DM' transverse to $(q_2 \circ p')^{-1}(L)$. Then by Proposition 1.2, $D\widetilde{M}$ is fibered, and the branched cover $q\colon D\widetilde{M} \to DM'$ takes fibers to fibers.

If $\widetilde{F}$ is the fiber surface of $D\widetilde{M}$, then q restricts on $\widetilde{F}$ to a twofold branched cover of a fiber surface of DM' transverse to $(q_2 \circ p')^{-1}(L)$, branched over their points of intersection. From Figure 7, we find 16 points of intersection between the arcs of $\pi((q_2 \circ p')^{-1}(L))$ and the bold arcs which determine the spinning curves. It follows that a spun fiber surface has 32 points of intersection with $(q_2 \circ p')^{-1}(L)$, since π maps these curves two-to-one. Since the spun fiber surface of DM' has genus two, an Euler characteristic calculation shows that $\widetilde{F}$ has genus 19. Together with the information in Lemma 3.3 and Proposition 3.4, this establishes the theorem. $\square$

Remark. For $n \geq 2$, the spun fiber surface for M' identified above pulls back to a fiber surface of genus $17n - 15$ in $D\widetilde{M}$, where $p\colon \widetilde{M} \to M_{2n,1}$ is the sixfold cover from the remark below Lemma 3.3.

4. The algebra of the tetrus

This section is devoted to proving the dichotomy of Proposition 0.3. Below we will use the *upper half space* model of hyperbolic space, $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}^+$, equipped with the complete Riemannian metric with all sectional curvatures -1. The orientation-preserving isometry group of this model is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$, acting extending its action on $(\mathbb{C} \times \{0\}) \cup \{\infty\}$, the *ideal boundary* of $\mathbb{H}^3$, by Möbius transformations. If an embedded totally geodesic plane has ideal boundary $L \cup \{\infty\}$, where L is a line in $\mathbb{C} \times \{0\}$, we will refer to it as the *hyperplane over* L.

Any complete orientable hyperbolic manifold M has associated to it a *Kleinian group* Γ, that is, a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$, with the property that M is isometric to $\mathbb{H}^3/\Gamma$. We will use the following characterization of arithmeticity for Kleinian groups, which can be found in [15, Theorem 8.3.2] for instance.

Proposition. *Let Γ be a finite-covolume Kleinian group. Then Γ is arithmetic if and only if the following three conditions all hold.*

(1) *The invariant trace field $k\Gamma$ has exactly one complex place.*
(2) *For each $\gamma \in \Gamma$, $\mathrm{tr}\,\gamma$ is an algebraic integer.*
(3) *The invariant quaternion algebra $A\Gamma$ is ramified at all real places of $k\Gamma$.*

Above, the invariant trace field $k\Gamma$ is the field obtained by adjoining to $\mathbb{Q}$ traces of elements of $\Gamma^{(2)} = \langle \gamma^2 \,|\, \gamma \in \Gamma \rangle < \Gamma$. The invariant quaternion algebra is the set of $k\Gamma$-linear combinations of elements of $\Gamma^{(2)}$, given the natural algebra structure it inherits from $M_2(\mathbb{C})$. The invariant quaternion algebra is *ramified* at a real place of $k\Gamma$ if after extending scalars to $\mathbb{R}$, it is isomorphic to the well known Hamilton's quaternions, the vector space over $\mathbb{R}$ with basis elements 1, i, j, and ij satisfying $i^2 = j^2 = (ij)^2 = -1$. See Sections 2.1 and 2.5 of [15], and Definition 3.3.6.

We note that a complete hyperbolic manifold M does not determine a unique Kleinian group Γ, as the quotient of $\mathbb{H}^3$ by Γ is isometric to its quotient by any conjugate of Γ, for example. However, it is easy to see that $k\Gamma$ is conjugation-invariant (since conjugation preserves the trace of an element of $\mathrm{PSL}_2(\mathbb{C})$), and that $A\Gamma$ is isomorphic to the invariant quaternion algebra of any conjugate of Γ. Mostow's rigidity theorem implies that the Kleinian group determined by a complete hyperbolic manifold M of finite volume is unique up to conjugation; hence $k\Gamma$ and $A\Gamma$ are invariants of M. It further holds, in fact, that $k\Gamma$ is a number field; that is, a finite extension of $\mathbb{Q}$, and that $k\Gamma$ and $A\Gamma$ are *commensurability* invariants of M, shared by all manifolds with which it has a common finite cover.

Although the definitions above do not give much clue as to the significance or relative plenitude of arithmetic hyperbolic manifolds, they have the advantage of being computable in several different ways. We refer the reader to [15] for a broad overview of the study of arithmetic hyperbolic 3-manifolds, and also for many of the computational shortcuts which we will use below.

In verifying Proposition 0.3 one may use SnapPea and Snap, programs which compute geometric and arithmetic invariants of hyperbolic 3-manifolds. The mathematics behind SnapPea is described in [11] and [27]. The program is available at `http://www.geometrygames.org/SnapPea`. Snap is introduced in [10], and is available at `http://www.ms.unimelb.edu.au/~snap`.

Since O_4 has underlying topological space B^3 and singular locus the tangle T of Figure 3, its double DO_4 has underlying topological space S^3 and singular locus the two-component link L on the left-hand side of Figure 1. We may thus regard DO_4 as obtained by $(4,0)$-Dehn surgery on L; that is, by removing the interior of each component of $\mathcal{N}(L)$ and, in its place, gluing a cylinder with conical singularity of angle $\pi/2$ so that the original meridian — the boundary slope of the four-holed sphere $\partial B^3 \cap \partial \mathcal{E}(T)$ — bounds a singular disk.

SnapPea describes a hyperbolic structure on the manifold obtained by $(4,0)$-surgery on each component of L, with hyperbolic volume approximately 5.724. Snap determines that the Kleinian group associated to DO_4 has a nonintegral trace and hence is non-arithmetic. Since $DM_{4,1}$ is a fourfold branched cover, its associated Kleinian group is a subgroup of the group DE_4 associated to DO_4. Thus since arithmeticity is a commensurability invariant, $DM_{4,1}$ is non-arithmetic.

The link L_μ on the right-hand side of Figure 1 is obtained from L by *mutation*. That is, L_μ is obtained from L by cutting along the sphere ∂B^3 which divides T from its mirror image and regluing by an order-two homeomorphism μ of ∂B^3 that preserves the set ∂T and acts on it as an even permutation. Theorem 2.6 of [22] asserts that a mutation homeomorphism of a four-punctured sphere in a hyperbolic manifold is realizable by an isometry, and it is well known that the same holds true for a sphere with four cone points of equal cone angle in a hyperbolic orbifold (the proof strategy outlined in [22, Remark 2.7] extends to this context, for example).

As evidence supporting the paragraph above, we note that SnapPea returns the same volume for the complement of L_μ as for that of L, and the same volume for their $(4,0)$-Dehn surgeries. In fact, we will explicitly identify the isometry realizing μ in Lemma 4.3 below. It follows that the orbifold which we will denote $D_\mu O_4$, resulting from $(4,0)$-Dehn surgery on each component of L_μ, is isometric to the *twisted double* of O_4 obtained by cutting DO_4 along the totally geodesic sphere ∂O_4 with four cone points and regluing by the isometry realizing μ.

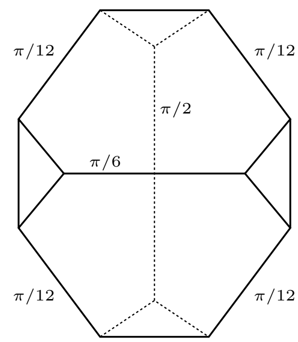

FIGURE 8. O_4 is a quotient of $\mathcal{T}_4$ above.

Applying Snap, we find that $D_\mu O_4$ is arithmetic. In fact, its invariant trace field and quaternion algebra are identical to those of DO_4 (this is a general fact about hyperbolic manifolds related by mutation; see [19]). The only difference in the invariant data is that the Kleinian group associated to $D_\mu O_4$ has integral traces.

It remains to show that μ lifts to an isometry of $\partial M_{4,1}$, so that there *exists* a manifold $D_\mu M_{4,1}$ which covers $D_\mu O_4$. This is equivalent to the assertion that a lift of μ to the universal cover normalizes the subgroup $\Lambda_4 < E_4$ corresponding to $\partial M_{4,1}$. Since we will explicitly describe such a lift in Lemma 4.3, we defer establishing this fact until then. Then $D_\mu M_{4,1}$ inherits arithmeticity from $D_\mu O_4$.

An alternative, direct approach to proving Proposition 0.3 uses explicit descriptions of Kleinian groups associated to DO_4 and $D_\mu O_4$. Let $\mathcal{T}_4$ be the truncated tetrahedron of Figure 8. (This is combinatorially isomorphic to the double of the partially truncated tetrahedron labeled $\mathcal{T}_n$ in [20, Figure 3], across its bottom face.) We will regard $\mathcal{T}_4$ as a polyhedron in $\mathbb{H}^3$, with the property that labeled edges have the specified dihedral angles and all others have dihedral angle π_2; in particular, each triangular face is perpendicular to the other faces it meets.

In [20, pp. 119–120], isometries h and x pairing the hexagonal faces of $\mathcal{T}_4$ are described, so that the quotient of $\mathcal{T}_4$ by this face pairing is O_4. The lemma below describes isometries H_4 and X_4 which realize these face pairings under the embedding of $\mathcal{T}_4$ determined by the following criteria: every $(z,t) \in \mathcal{T}_4$ has $\Im z \geq 0$, the bottom triangular face lies in the hyperplane over $\mathbb{R}$ with its vertex of dihedral angle $\pi/2$ at $(0,1)$, and $\mathcal{T}_4$ is preserved by reflection in the hyperplane over $i\mathbb{R}$.

Lemma 4.1. *Let* $\tau = \frac{1}{2\sqrt{2}}\left(\sqrt{3}+1+i\sqrt{4+6\sqrt{3}}\right)$, *let* $H_4 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$, *and let*

$$X_4 = \begin{pmatrix} \frac{\tau}{2}\left[1+\frac{\sqrt{3}+1}{3^{1/4}\sqrt{2}}\right] & -\frac{\tau}{2}\left[1+\frac{i\sqrt{4+6\sqrt{3}}}{3^{1/4}\sqrt{2}}\right]+\frac{1+\sqrt{3}}{\sqrt{2}} \\ \frac{\tau}{2}\left[1-\frac{i\sqrt{4+6\sqrt{3}}}{3^{1/4}\sqrt{2}}\right]-\frac{1+\sqrt{3}}{\sqrt{2}} & \frac{\tau}{2}\left[1-\frac{\sqrt{3}+1}{3^{1/4}\sqrt{2}}\right] \end{pmatrix}$$

There is an isometry taking O_4 to the convex core of $\mathbb{H}^3/E_4$, where $E_4 = \langle H_4, X_4 \rangle$ is as described in Theorem 2.1. The image of ∂O_4 is the quotient of the hyperplane $\mathcal{H}$ over $\mathbb{R}$ by its stabilizer in E_4, the subgroup $\Lambda_4 = \langle P_1, P_2, P_3\rangle$, where

$$P_1 = H_4^{-1} \qquad P_2 = H_4X_4H_4X_4^{-2} \qquad P_3 = (X_4H_4X_4)H_4^{-1}(X_4H_4X_4)^{-1}$$

Each of P_1, P_2, and P_3 is elliptic. The final conjugacy class of elliptic elements of Λ_4 is represented by $P_4 = P_1P_2P_3^{-1} = (X_4H_4X_4^{-1})H_4X_4H_4X_4^{-2}(X_4H_4X_4^{-1})^{-1}$.

Using a computer algebra program, one can check that E_4 as described above satisfies the relations of the presentation which we recorded in Theorem 2.1:

$$E_4 \cong \langle\, H_4, X_4 \mid (H_4)^4 = (H_4X_4H_4X_4^{-2})^4 = 1\,\rangle.$$

These relations are a consequence of Poincaré's polyhedron theorem. One may also verify, again by using a computer algebra program, that each of P_1, P_2, and P_3 is elliptic, of order 4, and contained in $\mathrm{PSL}_2(\mathbb{R})$. This fits with the orbifold surgery description of O_4 above, since the boundary subgroup Λ described in Lemma 2.1 is generated by parabolic elements corresponding to meridians of T.

Because ∂O_4 is the quotient of the hyperplane $\mathcal{H}$ over $\mathbb{R}$ by $\Lambda_4 = \mathrm{Stab}_{E_4}(\mathcal{H})$, $\mathcal{H}$ is a component of the boundary of the convex hull of the limit set of E_4. The Klein-Maskit combination theorem [16] thus yields a description of the Kleinian group corresponding to the double of O_4 across its boundary. Below, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{C})$, define $\bar{\gamma} = \left(\begin{smallmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{smallmatrix}\right)$.

Lemma 4.2. *Let $DE_4 = \langle E_4, \overline{E}_4\rangle$, where $\overline{E}_4 = \{\bar{\gamma} \mid \gamma \in E_4\}$. Then DO_4 is isometric to $\mathbb{H}^3/DE_4$.*

The fact that DE_4 has a nonintegral trace follows from a computation:

$$\mathrm{tr}\ X_4\overline{X}_4 = 2 + 5/\sqrt{3}.$$

This verifies the "non-arithmetic" half of Proposition 0.3. In order to verify the "arithmetic" half, we need to explicitly describe the Kleinian group corresponding to $D_\mu O_4$. The lemma below identifies the isometry which realizes μ.

Lemma 4.3. *Define*

$$M = \begin{pmatrix} \frac{\sqrt{\sqrt{3}+1}}{2} & \left(3^{1/4} + \frac{\sqrt{2}}{2}\right)\sqrt{\frac{\sqrt{3}+1}{2}} \\ -\left(3^{1/4} - \frac{\sqrt{2}}{2}\right)\sqrt{\frac{\sqrt{3}+1}{2}} & -\frac{\sqrt{\sqrt{3}+1}}{2} \end{pmatrix}$$

M is an involution normalizing Λ_4 and acting on the generators by

$$MP_1M^{-1} = P_2^{-1} \qquad MP_2M^{-1} = P_1^{-1} \qquad MP_3M^{-1} = P_1^{-1}P_4^{-1}P_1.$$

M induces self-isometries of ∂O_4 and $\partial M_{4,1}$, to which we will refer by μ (in both cases).

Proof. It is clear that M is an involution, since its trace is equal to 0; in fact, it is the 180-degree rotation in the midpoint of the geodesic arc in the hyperplane over $\mathbb{R}$ joining the fixed point of P_1 to that of P_2. Thus we expect it to exchange P_1 with $P_2^{\pm 1}$. That $MP_1M^{-1} = P_2^{-1}$ may be verified by direct computation, as may the action on P_3. Since any 3 elements of the collection $\{P_1, P_2, P_3, P_4\}$ generate Λ_4, it is normalized by M. Therefore M induces an isometry μ of ∂O_4.

We further recall, from Theorem 2.1, that the fundamental group $G_{4,1}$ of $M_{4,1}$ is $\ker \pi_1$, where $\pi_1 \colon E_4 \to \mathbb{Z}_4 = \langle h_4\rangle$ takes H_4 and X_4 to h_4. It follows from the description in Lemma 4.1 that $\pi_1(P_2) = \pi_1(P_4) = h_4$, and $\pi_1(P_1) = \pi_1(P_3) = h_4^{-1}$. Therefore, if $\phi_M \colon \Lambda_4 \to \Lambda_4$ is the automorphism induced by conjugation by M, we find that $\pi_1 \circ \phi_M = \iota \circ \pi_1$, where $\iota \colon \mathbb{Z}_4 \to \mathbb{Z}_4$ takes h_4 to h_4^{-1}. Since the identity subgroup is characteristic, ϕ_M preserves $\ker \pi_1|_{\Lambda_4}$. Hence M induces an isometry on $\partial M_{4,1}$, which corresponds to this subgroup. $\square$

The Klein–Maskit combination theorem now yields the description below.

Lemma 4.4. *Let $\mu\colon \partial O_4 \to \partial O_4$ be the isometry induced by M, and define $D_\mu O_4 = O_4 \cup_\mu \overline{O}_4$. There is an isometry from $D_\mu O_4$ to $\mathbb{H}^3/\langle E_4, M\overline{E}_4M^{-1}\rangle$.*

Below we describe the arithmetic data for $D_\mu O_4$.

Lemma 4.5. *Let $D_M E_4 = \langle E_4, M\overline{E}_4M^{-1}\rangle$. Then $D_M E_4$ has integral traces, the invariant trace field $kD_M E_4 = \mathbb{Q}(i\sqrt{4+6\sqrt{3}})$, and the invariant quaternion algebra has Hilbert symbol*

$$\left(\frac{-4, 2+4\sqrt{3}+i(3+\sqrt{3})\sqrt{4+6\sqrt{3}}}{kD_M E_4}\right).$$

In particular, $D_M E_4$ is arithmetic.

Proof. This proof is almost entirely computational, and we will spare the reader the details in favor of an overview. Using a presentation of $D_M E_4$ as a free product with amalgamation, it can be shown that it is generated by the elements H_4, X_4, and $\overline{X}_4^M \doteq M\overline{X}_4M^{-1}$. Then formula (3,26) and the remark below it in [15] imply that the trace of any element in $D_M E_4$ is an integral polynomial in the traces of the following elements.

$$H_4,\ X_4,\ \overline{X}_4^M,\ H_4X_4,\ H_4\overline{X}_4^M,\ X_4\overline{X}_4^M,\ H_4X_4\overline{X}_4^M$$

The invariant trace field is obtained by adjoining to $\mathbb{Q}$ the squares of the traces of the generators as well as the following products of traces, by [15, Lemma 3.5.9].

$$\begin{array}{ll} \mathrm{tr}H_4X_4\ \mathrm{tr}H_4\ \mathrm{tr}X_4 & \mathrm{tr}X_4\overline{X}_4^M\ \mathrm{tr}X_4\ \mathrm{tr}\overline{X}_4^M \\ \mathrm{tr}H_4\overline{X}_4^M\ \mathrm{tr}H_4\ \mathrm{tr}\overline{X}_4^M & \mathrm{tr}H_4X_4\overline{X}_4^M\ \mathrm{tr}H_4\ \mathrm{tr}X_4\ \mathrm{tr}\overline{X}_4^M \end{array}$$

Finally, the Hilbert symbol for the invariant quaternion algebra, which is determined by any nonelementary subgroup, may be obtained as

$$\left(\frac{\mathrm{tr}^2H_4(\mathrm{tr}^2H_4-4), \mathrm{tr}^2H_4\mathrm{tr}^2X_4(\mathrm{tr}[H_4,X_4]-2)}{kD_M E_4}\right),$$

according to [15, Theorem 3.6.2].

The formulae above yield the invariant data recorded in the statement of the lemma. The Galois conjugates of $i\sqrt{4+6\sqrt{3}}$, aside from its complex conjugate, are $\pm\sqrt{6\sqrt{3}-4}$; hence the invariant trace field has exactly one complex place. The invariant quaternion algebra is ramified at a real place of $kD_M E_4$ if and only if both entries of its Hilbert symbol are sent to negative numbers by the corresponding real embedding (see [15, Section 2.5]). This holds, and arithmeticity follows. $\square$

Remark. Snap outputs the following minimal polynomial for the invariant trace field $kDE_4 = kD_M E_4$:

$$x^4 - 2x^3 - x^2 + 2x - 2.$$

This factors over $\mathbb{Q}(\sqrt{3})$ as $(x^2 - x - 1 + \sqrt{3})(x^2 - x - 1 - \sqrt{3})$, after which an application of the quadratic formula yields the roots below.

$$\frac{1 \pm \sqrt{5+4\sqrt{3}}}{2} \qquad\qquad \frac{1 \pm \sqrt{5-4\sqrt{3}}}{2}$$

Since $5 - 4\sqrt{3} = -(4+6\sqrt{3})(1+\sqrt{3})^2$, Snap agrees with Lemma 4.5 regarding the invariant trace field.

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