

# Brownian Computation is Thermodynamically Irreversible

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Brownian computers are supposed to illustrate how logically reversible mathematical operations can be computed by physical processes that are thermodynamically reversible or nearly so. In fact, they are thermodynamically irreversible processes that are the analog of an uncontrolled expansion of a gas into a vacuum.

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## 1. Introduction

The thermodynamics of computation applies ideas from thermal and statistical physics to physical devices implementing computations. Its major focus has been to characterize the principled limits to thermal dissipation in these devices. The best case of no dissipation arises when we use processes that create no thermodynamic entropy. They are thermodynamically reversible processes in which all driving forces are in perfect balance.

Thermal fluctuations, such as arise through random molecular motions, are not normally a major consideration in thermodynamic analyses. However, they become decisive in the

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thermodynamics of computation. For the thermodynamic dissipation associated with thermodynamically irreversible processes is minimized by reducing the computational devices to the smallest scales possible, that is, to molecular scales. Thermal fluctuations now become a major obstacle to reducing thermodynamic dissipation. Consider a thermodynamic process, such as a single step of computation in a physical computer, at these molecular scales. In order to proceed to completion, it must overcome these fluctuations. The problem is serious. It is the essential idea behind a “no go” result described elsewhere (Norton, 2011, Section 7.4; forthcoming a; manuscript, Part II). If the process is to proceed to completion with reasonable probability, it follows quite generally that it must create thermodynamic entropy in excess of  $k \ln 2$  per step.

This quantity of entropy,  $k \ln 2$ , is the minimum amount associated by Landauer’s principle with the erasure of one bit of information. If each step of a computation *must* create more thermodynamic entropy than this Landauer limit, then any debate over the cogency of the Landauer principle is rendered superfluous. Indeed we have to give up the idea that the minimum thermodynamic dissipation is determined by the logical specification of the computation. For the minimum dissipation is fixed by the number of discrete steps in the computational procedure used, which makes this minimum dependent on the implementation.

The no go result wreaks greatest harm when the computer proceeds with what I shall call a “discrete protocol” tacitly presumed above. It is the familiar protocol in which the computation is divided into a series of discrete steps, each of which must be completed before the next is initiated.

There is an escape from the no go result. Bennett (1973, 1982) and Bennett and Landauer (1985) have described a most ingenious protocol for computation that minimizes its effects. In the protocol, called “Brownian” computation, the many logical steps of a complicated computation are collapsed into a single process thermodynamically. It is done by chaining the logical steps of the computation into a single process such that random thermal motions carry the computational device’s state back and forth over the steps in a way that is analogous to the Brownian motion of a pollen grain in water. No step is assuredly complete until the device happens to enter a final, dissipative trap state from which it escapes with very low probability.

The no go result still applies to this new, indiscrete protocol, but now the thermodynamic entropy creation required is merely that required for one step. It can be negligible in the context

of a large and complicated computation if that single step really is close to thermodynamic reversibility. That is the hope. However, it is not realized.

For all the mechanical and computational ingenuity of the devices, the thermodynamic analysis Bennett provides is erroneous. The devices are described as implementing thermodynamically reversible computations, or coming close to it, thereby demonstrating the possibility in principle of thermodynamically reversible computation. In fact the devices are thermodynamically irreversible. They implement processes that are the thermodynamic analog of an uncontrolled, irreversible expansion of a one-molecule gas, the popping of a balloon of gas into a vacuum.

Sections 2 and 3 below will describe the operation of a Brownian computer and give a thermodynamic analysis of it. The main result is that an  $n$  stage computation creates  $k \ln n$  of thermodynamic entropy; and that extra thermodynamic entropy is created if a trap state is introduced to assure termination of the computation; or if an energy gradient is introduced to speed up the computation.

Section 4 affirms the main claim of this paper, that, contrary to the view in the literature, Brownian computation is thermodynamically irreversible. Section 5 reviews several ways that one might come to misidentify a thermodynamically irreversible process as reversible. The most important is the practice in the thermodynamics of computation of tracking energy instead of entropy in an effort to gauge which processes are thermodynamically reversible.

Finally, if a Brownian computer implements logically irreversible operations, its accessible phase space may become exponentially branched. This branching has been associated with Landauer's principle of the necessity of an entropy cost of erasure. In Section 6, it is argued that the connection is spurious and that Brownian computation can provide no support for the supposed minimum to the entropy cost. Brownian computation is powered by a thermodynamically irreversible creation of entropy and it creates thermodynamic entropy whether it is computing a logically reversible or a logically irreversible operation. It cannot tell us what the minimum dissipation must be if we were to try to carry out the same operations with thermodynamically reversible processes.

## 2. Brownian Computers

All bodies in thermal contact with their environment exhibit fluctuations in their physical properties. They are indiscernible in macroscopic bodies. Fluctuation driven motions are visible through an optical microscope among tiny particles suspended in water. The botanist Robert Brown observed them in 1827 as the jiggling of pollen grains, but he did not explain them. In his year of miracles of 1905, Einstein accounted for the motions as thermal fluctuations. When we proceed to still smaller molecular scales, these thermal motions become more important. In biological cells they can bring reagents into contact and are involved in the complicated chemistry of DNA and RNA. Bennett, sometimes in collaboration with Landauer (Bennett, 1973, 1982; Bennett and Landauer 1985), notes that the molecular structures involved with DNA and RNA are at a level of complexity that they could be used to build computing devices whose function would, in some measure, be dependent on the thermal motions of the reagents. They then develop and idealize the idea as the notion of a mechanical computing device powered by these random thermal motions. These are the Brownian computers.

To see how these thermal motions can have a directed effect, consider the simplest case of a small particle released in the leftmost portion a long channel, shown from overhead in Figure 1. Random thermal motions will carry the particle back and forth in the familiar random walk. If a low energy trap is located at the rightmost end of the channel, the particle will eventually end up in it. It will remain there with high probability, if the trap is deep enough.

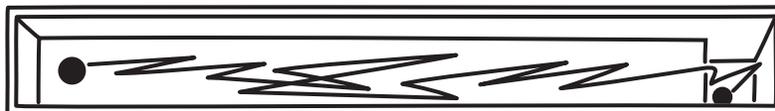


Figure 1. Brownian motion of particle in a channel

Bennett suggests that this sort of motion can drive forward a vastly more complicated contrivance of many mechanical parts that implements a Turing machine and hence carries out computations. It consists of many interlocked parts that can slide over one another. The continuing thermal jiggling of the parts leads the device to meander back and forth between the many states that comprise the steps of the computation.

The reader is urged to consult the works cited above for drawings and a more complete description of the implementation of the Brownian computer.

The computer must be assembled from rigid components that interlock and slide over one another. It consists of various shapes that can slide up and down from their reference position to function as memory storage devices; actuator rods that move them; rotating disks with grooves in them to move the actuators; and so on. No friction is allowed, since that would be thermodynamically dissipative; and no springs are allowed. A spring-loaded locking pin, for example, would fail to function. Once the spring drives the pin home, it would immediately bounce out because of the time-reversible, non-dissipative dynamics assumed.

While Bennett's accounts describe many essential parts of the Brownian computer, many more are not described. No doubt, a complete specification of all the parts of the Brownian computer would be lengthy. However, without it, we must assume with Bennett that the device really can be constructed from the very limited repertoire of processes allowed. That is, the possibility of the device and thus the entire analysis remains an unproven conjecture. I will leave the matter open since there are demonstrable failures in the analysis to be elaborated below, even if the conjecture is granted.

For reasons that will be apparent later, Bennett mostly considers Brownian computations in which each computational state has a unique antecedent state. This condition is met if the device computes only logically reversible operations, such as NOT. For then, if the present state of a memory cell is 0, its antecedent state must have been 1; and vice versa. However the condition is not realized if the device computes logically irreversible operations, such as the erase function. For then, if the present state of a memory cell is the erasure value 0, its antecedent state may have been either a 0 or a 1.

That each state has a unique antecedent state requires that the whole device implement a vastly complicated system of interlockings, so that the entire device has only one degree of freedom. The computation is carried out by the device meandering along this one degree of freedom. The effect of this requirement, as implemented by Bennett, has an important abstract expression. The position and orientation of each component of the massively complicated Brownian computer can be specified by their coordinates. The combination of them all produces a configuration space of very high dimension. The limitation to a single degree of freedom results in the accessible portion of the configuration space being a long, labyrinthine, one-dimensional channel with a slight thickness given by the free play of the components.

Figure 2 illustrates how this channel comes about in the simplest case of two components constrained to move together. The components are bar and a plate with a diagonal slot cut into it. The bar has a pin fixed to its midpoint and the pin engages with the slot in the plate. Without the pin, the two components would be able to slide independently with the two degrees of freedom labeled by  $x$  and  $y$ . The confinement of the pin to the slot constrains them to move together, reducing the possible motions to a single degree of freedom. That single degree of freedom corresponds to the diagonal channel in their configuration space shown at right.

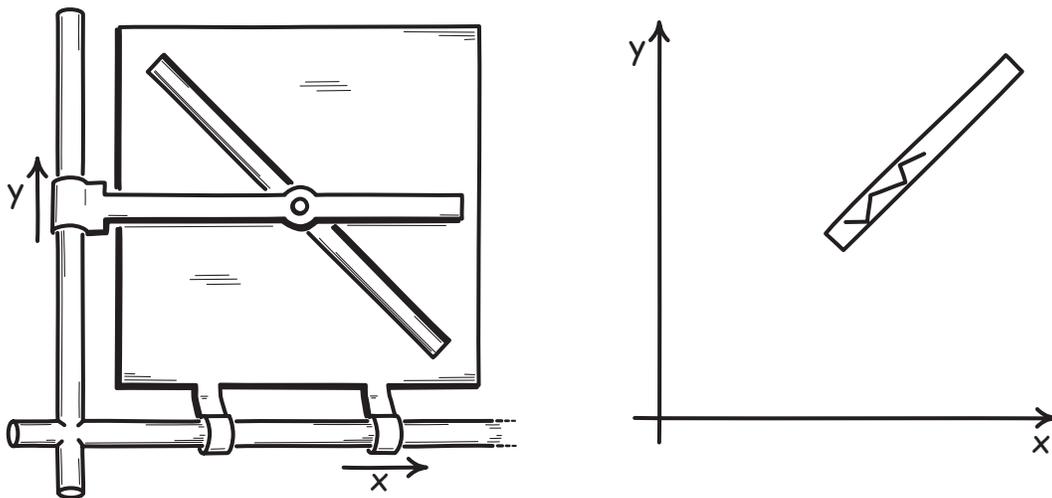


Figure 2. Two Components with a Single Common Degree of Freedom.

The channel in the configuration space of a Brownian computer would be vastly more complicated. It will end with a low energy trap analogous to the one shown in Figure 1 so that the computation is completed with high probability.

Here is Bennett's (1984) brief summary:

In a Brownian computer, such as Bennett's enzymatic computer, the interactions among the parts create an intricate but unbranched valley on the many-body potential-energy surface, isomorphic to the desired computation, down which the system passively diffuses, with a drift velocity proportional to the driving force.

The summary includes an unneeded complication. Bennett presumes that some slight energy gradient is needed to provide a driving force that will bring the computation towards its end state. In fact, as we shall see shortly, entropic forces are sufficient, if slower.

### 3. Thermodynamic Analysis of Brownian Computers

Bennett and Landauer (Bennett, 1973, 1982; Bennett and Landauer 1985) report several results concerning the thermodynamic and stochastic properties of Brownian computers. They do not provide the computations needed to arrive at the results. They are, apparently, left as an exercise for the reader. In this section, I will do the exercise. As we shall see in this and the following sections, I am able to recover some of the results concerning probabilities. However the fundamental claim that the Brownian computer operates at or near thermodynamic reversibility will prove unsustainable.

#### 3.1 Uncontrolled Expansion of a Single Molecule Gas

As a warm-up, I will recall the thermodynamics of the  $n$ -fold expansion of a single molecule ideal gas at temperature  $T$ . The gas molecule is initially trapped by a partition in the first cell of volume  $V$  of a long chamber of volume  $nV$ , as shown in Figure 3. The partition is removed and the gas expands irreversibly into the larger volume  $nV$ .

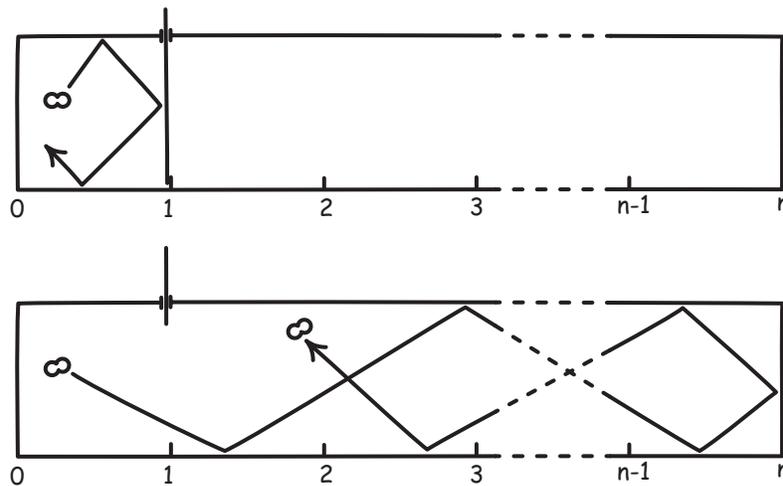


Figure 3. Irreversible Expansion of a one molecule gas

The Hamiltonian of the single molecule is given by

$$H = \pi(\mathbf{p}) \tag{1}$$

in the region of space accessible to the gas and infinite elsewhere. Here  $\mathbf{p}$  is a vector representing the momentum degrees of freedom of the molecule and  $\pi$  is some function of them, typically quadratic. The key point to note is that the Hamiltonian  $H$  is not a function of the spatial

coordinates  $\mathbf{x} = (x, y, z)$  of the molecule. This independence drives the results that follow. We assume that the  $x$  coordinate is aligned with the long axis of the chamber and that it has a cross-sectional area  $A$ .

At thermal equilibrium, the molecule's position is Boltzmann distributed probabilistically over its phase space as

$$p(\mathbf{x}, \mathbf{p}) = \exp(-H/kT)/Z(\lambda) \quad (2)$$

where we assume the molecule is confined to a region  $x=0$  to  $x=\lambda$  of the chamber. Using  $V=A\lambda$ , the partition function  $Z(\lambda)$  associated with the molecule confined to the region  $x=0$  to  $x=\lambda$  is

$$Z(\lambda) = \int_{all\ p} \int_{x=0}^{\lambda} \exp(-H/kT) A dx dp = V \lambda \int_{all\ p} \exp(-\pi(\mathbf{p})/kT) d\mathbf{p} \quad (3)$$

The associated canonical thermodynamic entropy is

$$S(\lambda) = \frac{\partial}{\partial T} (kT \ln Z(\lambda)) = k \ln(V\lambda) + S_p(T) \quad (4)$$

The contribution of the momentum degrees of freedom is absorbed into a constant  $S_p(T)$  that will not figure in the subsequent calculations. The independence of the Hamiltonian (1) from the position coordinates leads to the characteristic logarithmic volume dependence of the canonical entropy (4), that is, that  $S(\lambda)$  varies as  $k \ln(V\lambda)$ .

It follows that the thermodynamically irreversible  $n$ -fold increase in volume of the one molecule gas from  $\lambda=1$  to  $\lambda=n$  is associated with an entropy change

$$\Delta S_{\text{gas}} = k \ln(Vn) - k \ln V = k \ln n \quad (5)$$

During the expansion, the mean energy of the gas remains constant and, since it does no work, no net heat is exchanged with the environment. Since the environment is unchanged, we have for its thermodynamic entropy change

$$\Delta S_{\text{env}} = 0 \quad (6)$$

Thus the total entropy change is

$$\Delta S_{\text{tot}} = \Delta S_{\text{gas}} + \Delta S_{\text{env}} = k \ln n \quad (7)$$

Since the internal energy  $E$  remains the same, it follows from (5) that the change in free energy  $F = E - TS$  of the gas is

$$\Delta F_{\text{gas}} = -kT \ln n \quad (8)$$

We recover the same result from (3) and the canonical expression  $F = -kT \ln Z$ .

The essential point for what follows is that this expansion is driven entirely by entropic forces. There is no energy gradient driving it; the internal energy  $E$  of the gas is the same at the start of the expansion, when it is confined to volume  $V$ , as at the end, when it occupies a volume  $nV$ .

More generally, this sort of process is driven by an imbalance of a generalized thermodynamic force. For isothermal processes whose stages are parameterized by  $\lambda$ , the appropriate generalized force is

$$X = - (\partial F / \partial \lambda) \quad (9)$$

If we parameterize the states of the isothermally expanding one molecule gas by the volume  $V(\lambda) = V\lambda$ , occupied at stage  $\lambda$ , then  $F(\lambda) = -kT \ln V(\lambda)$  and the generalized force adopts the familiar form of the pressure of a single-molecule ideal gas:

$$X = T \frac{\partial k \ln V(\lambda)}{\partial V(\lambda)} = \frac{kT}{V(\lambda)} \quad (10)$$

### 3.2 Brownian Motion

One of the papers of Einstein's *annus mirabilis* of 1905 gives his analysis of Brownian motion (Einstein, 1905). In the paper he noted that the thermal motions of small particles suspended in a liquid would be observable under a microscope and he conjectured that their motions were the same as those observed in pollen grains by the botanist Brown. Einstein's goal was to give an account of these thermal motions within the molecular-kinetic theory of heat and thereby finally to establish it as the correct account of thermal processes.<sup>2</sup>

His starting point was to propose the astonishing idea that, from the perspective of the molecular-kinetic theory, individual molecules and microscopically visible particles can be treated by the same analysis and will give the same results. To reflect this astonishing idea, the analysis just given above of the statistical physics of a single molecule, has been written in such a way that it can be applied without change to a microscopically visible particle, such as a pollen grain. The controlling fact is that the Hamiltonian for a microscopically visible particle can be written as (1), for the energy of the particle will be independent of its position in the suspending liquid. The particular expression  $\pi(\mathbf{p})$ , which gives the dependence of the Hamiltonian on the

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<sup>2</sup> For an account of Einstein analysis, see Norton (2006, Section 3).

momentum degrees of freedom, will be different. For the particle is, to first approximation, moving through a resisting, viscous medium. However this difference will not affect the results derived above.

First, we will be able to conclude that a single Brownian particle will exert a pressure conforming to the ideal gas law, as shown in (10). What this means is that the collisions of the Brownian particle with the walls confining it to some volume  $V$  will lead to a mean pressure equal to  $kT/V$  on the walls. Einstein considered the case of the confining walls as a semi-permeable membrane that allows the liquid but not the particle to pass. Then the pressure is appropriately characterized as an osmotic pressure.

Second, the volume dependence of the thermodynamic entropy of the Brownian particle will conform to (4), so that an  $n$ -fold expansion of the volume accessible to the particle will be associated with an increase of thermodynamic entropy of  $\Delta S = k \ln n$  as shown in (5). By the same reasoning as in the case of the one molecule gas, the increase in total entropy is also  $\Delta S_{\text{tot}} = k \ln n$  as given by (7).

In direct analogy with the irreversible expansion described above for a single molecule gas, we can form a liquid filled chamber of volume  $nV$  with the Brownian particle trapped by a partition in the leftmost volume  $V$ , as shown in Figure 4. The particle exerts a pressure on the partition of  $kT/V$ . When the partition is removed, the unopposed pressure will lead to a thermodynamically irreversible expansion of the one Brownian particle gas into the full chamber. The uncontrolled expansion from volume  $V$  to  $nV$  is associated with the creation of  $k \ln n$  of thermodynamic entropy.

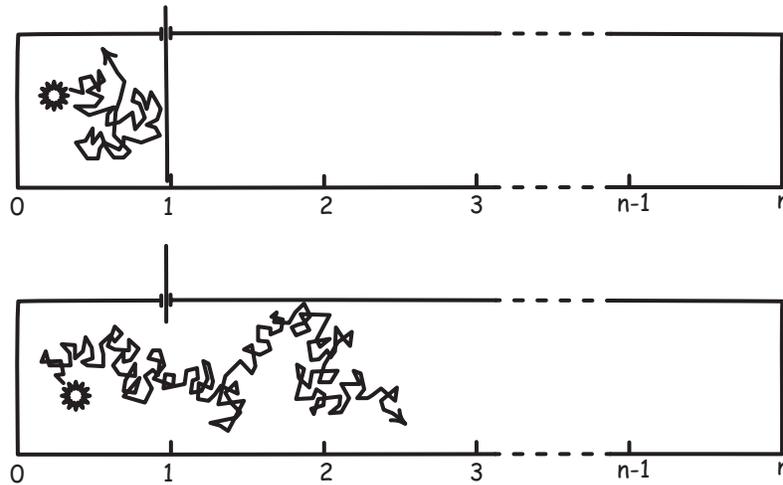


Figure 4. Irreversible Expansion of a one Brownian particle gas

Thermodynamically, the expansion of the one molecule gas and the one Brownian particle gas are the same. The two Figures 3 and 4, however, suggest the great dynamical differences. Ordinary gas molecules at normal temperatures move quickly, typically at many hundreds of meters per second. The motion of the one molecule is unimpeded by any other molecules, so it moves freely between the collisions with the walls. Brownian particles have the same mean thermal energy of  $kT/2$  per degree of freedom. But since they are much more massive than molecules, their motion is correspondingly slower. More importantly, they undergo very many collisions: the jiggling motion of a pollen grain visible under a microscope is the resultant of enormously many collisions with individual water molecules in each second.

This means that the expansion of the one Brownian particle gas is very much slower than that of the one molecule gas. When we observe the Brownian particle under the microscope, we are watching it for the briefest moment of time if we set our time scales according to how long the particle will take to explore the volume accessible to it. If we were to watch it for an extended time, we would see that the particle has adopted a new equilibrium state in which it explores the full volume  $nV$ , just as the expanded one molecule gas explores the same volume  $nV$ .

These differences of time scales between the one molecule gas and the one Brownian particle gas are irrelevant, however, to the thermal equilibrium states. Both gases start out in an equilibrium state confined to a volume  $V$ ; they undergo an uncontrolled,  $n$ -fold expansion to a new equilibrium state confined to volume  $nV$ ; and their thermodynamic entropies each increase by  $k \ln n$ .

These remarks draw on the analysis of the earlier parts of Einstein's (1905) paper. In sections 3 and later, he took up another aspect of Brownian motion that will not arise in the otherwise analogous physics of Brownian computers. Einstein modeled the Brownian particles as spheres and the surrounding water as a viscous fluid. (There is no analog of the fluid in the Brownian computer.) Einstein then modeled the diffusion of Brownian particles through the liquid as governed by the balance of two forces: the driving force of osmotic pressure in a gradient of particles and the opposing viscous forces as the particles move. What matters for our purposes is that Einstein eventually arrived at a result in the new theory of stochastic processes being created by his paper that is more general than the particular case he analyzed.

It is a result concerning particles, such as Brownian particles, that are animated in a random walk. Their positions spread through space according to a Gaussian distribution whose spatial variance is proportional to time. It follows that the average (absolute) distance  $d(t)$  covered in some time  $t$  is proportional to  $\sqrt{t}$ . This means that we cannot speak meaningfully of the average speed over time of the Brownian particle, for that average speed

$$d(t)/t \text{ is proportional to } 1/\sqrt{t} \rightarrow 0 \text{ as } t \rightarrow \infty \quad (11)$$

That is, if one tries to estimate average speed by forming the familiar ratio “distance/time,” that ratio can be made arbitrarily small by allowing time to become arbitrarily large. Einstein (1907, p. 42) remarks that the “...speed thus provided corresponds to no objective property of the motion investigated...”

### ***3.3 The Undriven Brownian Computer without Trap***

A Brownian computer behaves thermodynamically like a one molecule gas or a one Brownian particle gas expanding irreversibly into its configuration space. Here I will develop the simplest case of the undriven Brownian computer without a trap. This is the case that is closest to the irreversible expansion of a one molecule/Brownian particle gas. While it does not terminate the computation usefully, it sets the minimum thermodynamic entropy creation for all Brownian computers. Later we will add extra processes, such as a slight energy gradient to drive the computation faster, or an energy trap to terminate it. Each of these additions will create further thermodynamic entropy.

In this simplest case, the Brownian computer explores a one-dimensional labyrinthine channel in its phase space. All spatial configurations in the channel are assumed to have the same energy; there is no energy gradient pressing the system in one or other direction. As a result, the Hamiltonian of the Brownian computer is of the form (1), where we have the controlling fact that it depends only on the momentum degrees of freedom. The analysis proceeds as before.

We divide the very high dimensional configuration space of the computer into  $n$  stages. Precisely how the division is effected will depend upon the details of the implementation. One stage may correspond to all configurations in which the Turing machine reader head is interacting directly with one particular tape cell. For simplicity, we will assume that each stage occupies the same volume  $V$  in configuration space. Progress through the channel is parametrized by  $\lambda$ , which counts off the stages passed.

To operate the computer, its state is localized initially in the volume of phase space corresponding to the first stage,  $\lambda=0$  to  $\lambda=1$ . The device is then unlocked—the thermodynamic equivalent of removing the partition in the gas case—and the computer undertakes a random walk through the accessible channel in its phase space. As with the one molecule gas, changes in the momentum degrees of freedom play no role in the expansion. The computer settles down into a new equilibrium state in which it explores the full volume  $nV$  of the channel of its configuration space. The expansion is driven by an unopposed generalized force  $X$  given by (9), and with a volume dependence in configuration space of the single-molecule ideal gas law (10)

The expansion is illustrated in Figure 5, which also shows the constant energy dependence of the computer on the configuration space.

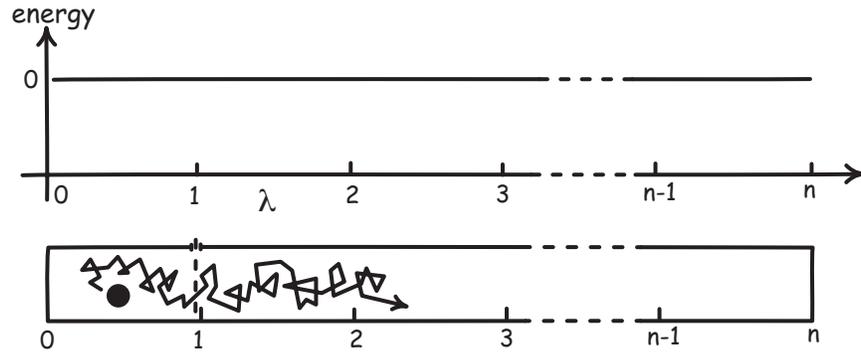


Figure 5. Undriven Brownian computer without Trap

We arrive at two results. First, since the Hamiltonian is independent of the spatial configuration in the accessible channel, it follows from the Boltzmann distribution (2) that the computer's state is distributed uniformly over the  $n$  stages of  $\lambda$ . That is, its probability density is

$$p(\lambda) = 1/n \quad (12a)$$

The thermodynamically irreversible, uncontrolled  $n$ -fold expansion will create thermodynamic entropy

$$\Delta S_{\text{comp}} = k \ln n \quad (5a)$$

The environment will be unaffected; no work is done by the expansion and no net heat passes between the environment and the computer:

$$\Delta S_{\text{env}} = 0 \quad (6a)$$

Thus the total thermodynamic entropy change is

$$\Delta S_{\text{tot}} = \Delta S_{\text{comp}} + \Delta S_{\text{env}} = k \ln n \quad (7a)$$

This is the minimum thermodynamic entropy creation associated with the operation of the Brownian computer. Embellished versions below add processes that create more thermodynamic entropy.

As before, the free energy change is

$$\Delta F_{\text{comp}} = -kT \ln n \quad (8a)$$

### 3.4 The Undriven Brownian Computer with Trap

This last Brownian computer is not useful for computation since its final, equilibrium state is uniformly distributed over all stages of the computation. The remedy is to add an extra stage,  $\lambda=n$  to  $\lambda=n+1$ , in which the computer's energy is dependent on the spatial positions of its parts, that is, on its position in configuration space. In the final trap stage, the energy of the system will be  $E_{\text{trap}}$  less than the position independent energy of the other stages, which are set by convention to 0. This trapping energy is set so that occupation of the final trap stage is probabilistically preferred to whichever extent we choose. When the computer moves into this final trap state, the computer state corresponds to that of completion of the computation. This is illustrated in Figure 6.

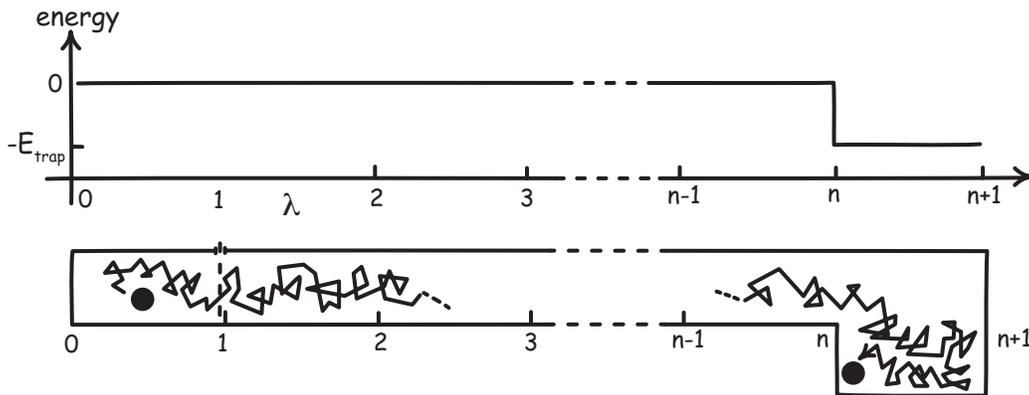


Figure 6. Undriven Brownian Computer with a Low Energy Trap

The addition of the energy trap introduces a configuration space dependence of the Hamiltonian. Within the accessible channel, it is now

$$H = \pi(\mathbf{p}) + \varphi(\lambda) \quad (1b)$$

where  $\varphi(\lambda) = 0$  for  $0 < \lambda < n$  and  $\varphi(\lambda) = -E_{\text{trap}}$  for  $n < \lambda < n+1$ . The Boltzmann distribution becomes

$$p(\lambda, \mathbf{p}) = \exp(-H/kT)/Z(n+1) \quad (2b)$$

where the partition function is

$$\begin{aligned} Z(n+1) &= \int \exp(-H/kT) dx dp = \int_{all\ p} \exp(-\pi(\mathbf{p})/kT) dp \cdot \int_{\lambda=0}^{n+1} \exp(-\varphi(\lambda)/kT) V d\lambda \\ &= M \cdot V \cdot (n + \exp(E_{trap}/kT)) \end{aligned} \quad (3b)$$

using the fact that the volume element of configuration space  $dx = V d\lambda$  and writing the contribution from the momentum degrees of freedom as  $M = \int_{all\ p} \exp(-\pi(\mathbf{p})/kT) dp$ . Since the momentum degrees of freedom are uninteresting, we integrate them out and recover the probability densities

$$p(\lambda) = \frac{1}{n + \exp(E_{trap}/kT)} \text{ for } 0 < \lambda < n \quad p(\lambda) = \frac{\exp(E_{trap}/kT)}{n + \exp(E_{trap}/kT)} \text{ for } n < \lambda < n+1 \quad (12b)$$

It follows that the probability P that the computer is in the trap state  $n < \lambda < n+1$  is

$$P = 1/(1 + n \cdot \exp(-E_{trap}/kT)) \quad O_P = \exp(E_{trap}/kT)/n$$

where  $O_P = P/(1-P)$  is the odds of the computer being in the final trap state. Inverting this last expression enables us to determine how large the trapping energy  $E_{trap}$  should be for any nominated P or  $O_P$ :

$$E_{trap} = kT(\ln n + \ln O_P) = kT \ln n + kT \ln (P/(1-P)) \quad (13b)$$

We compute the thermodynamic entropy of the expanded equilibrium state as

$$\begin{aligned} S(n+1) &= \frac{\partial}{\partial T} (kT \ln Z(n+1)) = \frac{\partial}{\partial T} (kT \ln M) + \frac{\partial}{\partial T} (kT \ln V) + \frac{\partial}{\partial T} (kT \ln(n + \exp(E_{trap}/kT))) \\ &= S_p(T) + k \ln V + k \ln(n + \exp(E_{trap}/kT)) - P \cdot E_{trap}/T \end{aligned} \quad (4b)$$

The first term  $S_p(T)$  represents the contribution of the momentum degrees of freedom and is independent of stage of computation achieved. Hence, as before, it need not be evaluated more specifically.

The thermodynamic entropy of the initial state is  $S(1) = S_p(T) + k \ln V$  as before.

Therefore, the increase in thermodynamic entropy in the course of the thermodynamically irreversible expansion and trapping of the computer state is

$$\Delta S_{comp} = k \ln (n + \exp(E_{trap}/kT)) - P \cdot E_{trap}/T \quad (5b)$$

In the course of the thermodynamically irreversible expansion, when the system falls into the final energy trap, it will release energy  $E_{trap}$  as heat to the environment. More carefully, on

average it will release energy  $P \cdot E_{\text{trap}}$  since the computer state will only be in the trap with high probability  $P$ . This will increase the thermodynamic entropy of the environment by<sup>3</sup>

$$\Delta S_{\text{env}} = P \cdot E_{\text{trap}}/T \quad (6b)$$

Thus the total thermodynamic entropy change is

$$\begin{aligned} \Delta S_{\text{tot}} &= \Delta S_{\text{comp}} + \Delta S_{\text{env}} = k \ln (n + \exp(E_{\text{trap}}/kT)) \\ &= k \ln n + k \ln (1 + O_P) \end{aligned} \quad (7b)$$

Hence the effect of adding the trap is to increase the net creation of thermodynamic entropy over that of the untrapped system (7a) by the second term  $k \ln (1 + O_P) = k \ln (1/(1-P))$ . The added term will be larger according to how much we would like the trap state to be favored, that is, how large we set the odds  $O_P$ .

Rearranging (5b), we find that change in free energy  $F=E-TS$  is

$$\Delta F_{\text{comp}} = -kT \ln n - kT \ln (1 + O_P) \quad (8b)$$

We recover the same result from (3b) and the canonical expression  $F = -kT \ln Z$ .

### ***3.5 The Energy Driven Brownian Computer without Trap***

This last case of the undriven but trapped Brownian computer is sufficient to operate a Brownian computer. Bennett (1973, p. 531; 1982, p. 921), however, includes the complication of a slight energy gradient in the course of the computation, in order to speed up the computation. We can understand the thermodynamic import of this augmentation by considering the simpler case of an energy gradient driven computer, without the energy trap.

The energy gradient is included by assuming that there is linear spatial dependence of the energy of the system on the parameter  $\lambda$  that tracks progress through the accessible channel in the phase space. That is, we assume an energy ramp of  $\epsilon$  per stage. The Hamiltonian becomes:

$$H = \pi(\mathbf{p}) - \epsilon\lambda \quad (1c)$$

This is illustrated in Figure 7.

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<sup>3</sup> While the process is not thermodynamically reversible, we recover the same thermodynamic entropy change for the environment by imagining another thermodynamically reversible process in which heat energy  $P \cdot E_{\text{trap}}$  is passed to the environment.

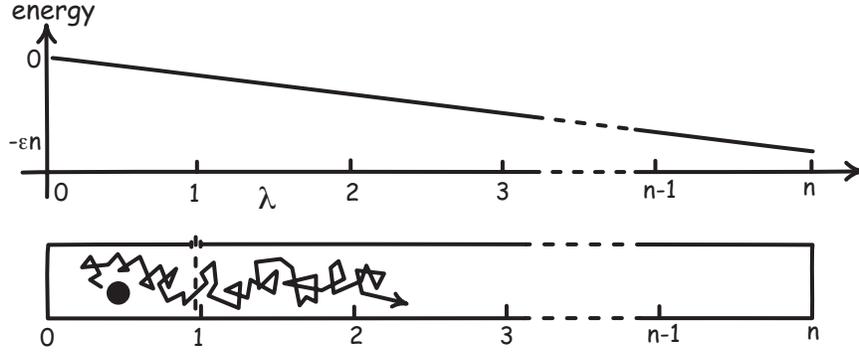


Figure 7. Energy Driven Brownian Computer without Trap

The effect of the energy ramp will be to accelerate progress towards the completion of the computation as well as skewing the equilibrium probability distribution towards the final stage. It will, however, prove to be a thermodynamically inefficient way of assuring completion. That assurance is achieved more efficiently with an energy trap, as I believe is Bennett's intent.

As before, the probability density over the  $n$  stages of the computation is

$$p(\lambda, \mathbf{p}) = \exp(-H/kT)/Z(n) \quad (2c)$$

where the partition function is

$$\begin{aligned} Z(n) &= \int \exp(-H/kT) dx dp = \int_{\text{all } p} \exp(-\pi(\mathbf{p})/kT) dp \cdot \int_{\lambda=0}^n \exp(\epsilon\lambda/kT) V d\lambda \\ &= M \cdot V \cdot (kT/\epsilon) \cdot (\exp(\epsilon n/kT) - 1) \end{aligned} \quad (3c)$$

As before,  $M$  represents the contribution of momentum degrees of freedom. Integrating out these momentum degrees of freedom, we recover the probability density for the accessible channel of configuration space

$$p(\lambda) = \frac{\epsilon}{kT} \frac{\exp(\epsilon\lambda/kT)}{\exp(\epsilon n/kT) - 1} \quad (12c)$$

Hence the probability that the computer state is in the final stage is

$$P = \int_{n-1}^n p(\lambda) d\lambda = \frac{\exp(\epsilon n/kT) - \exp(\epsilon(n-1)/kT)}{\exp(\epsilon n/kT) - 1} \approx 1 - \exp(-\epsilon/kT)$$

where the approximation is that  $\epsilon n/kT \gg 1$ . Inverting, we find that the energy gradient  $\epsilon$  per stage to achieve a final stage probability  $P$  is

$$\epsilon = kT \ln(1/(1-P))$$

For desirable values of P that are close to unity, this last formula shows that a steep energy gradient is needed. For a P = 0.99, we would require  $\epsilon = kT \ln(100) = 4.6 kT$ . Finally, If we assume in addition that  $\epsilon \ll kT$ , this probability reduces to

$$P = 1 - \exp(-\epsilon/kT) \approx \epsilon/kT,$$

This conforms with Bennett's (1973, p.51) remark that:

...if the driving force  $\epsilon$  is less than  $kT$ , any Brownian computer will at equilibrium spend most of its time in the last few predecessors of the final state, spending about  $\epsilon/kT$  of its time in the final state itself

Before computing the thermodynamic entropy change, it will be convenient to compute the mean energy of the initial state  $E(1)$  and the final state  $E(n)$  associated with the configuration space degrees of freedom. We have for the mean energy that

$$\begin{aligned} E(n) &= kT^2 \frac{\partial}{\partial T} \ln Z(n) \\ &= kT^2 \frac{\partial}{\partial T} \ln M + kT^2 \frac{\partial}{\partial T} \ln(kT/\epsilon) + kT^2 \frac{\partial}{\partial T} \ln(\exp(\epsilon n/kT) - 1) \\ &= E_p(T) + kT - \frac{\epsilon n}{1 - \exp(-\epsilon n/kT)} \end{aligned}$$

where  $E_p(T)$  represents the contribution of the momentum degrees of freedom and is independent of stage of computation achieved. Setting  $n=1$ , we find

$$E(1) = E_p(T) + kT - \frac{\epsilon}{1 - \exp(-\epsilon/kT)}$$

We now compute the thermodynamic entropy of the final equilibrium state as<sup>4</sup>

$$\begin{aligned} S(n) &= \frac{\partial}{\partial T} (kT \ln Z(n)) \\ &= \frac{\partial}{\partial T} (kT \ln M) + \frac{\partial}{\partial T} (kT \ln V) + \frac{\partial}{\partial T} (kT \ln(kT/\epsilon)) + \frac{\partial}{\partial T} (kT \ln(\exp(\epsilon n/kT) - 1)) \\ &= S_p(T) - E_p(T)/T + k \ln V + k \ln(kT/\epsilon) + k \ln(\exp(\epsilon n/kT) - 1) + E(n)/T \\ &= k \ln V + k \ln(kT/\epsilon) + k \ln(\exp(\epsilon n/kT) - 1) + E(n)/T \end{aligned} \tag{4c}$$

---

<sup>4</sup> The expression is simplified using  $S_p(T) = E_p(T)/T$ . This follows from considering the momentum degrees of freedom contribution to both entropy and energy during a thermodynamically reversible heating from  $T=0$ .

As before,  $S_p(T)$  represents the contribution of the momentum degrees of freedom and is independent of stage of computation achieved. Setting  $n=1$ , we find the initial state thermodynamic entropy to be

$$S(1) = k \ln V + k \ln(kT / \varepsilon) + k \ln(\exp(\varepsilon / kT) - 1) + E(1)/T$$

The thermodynamic entropy increase of the Brownian computer during its irreversible expansion from stage 1 to stage  $n$  is given by

$$\begin{aligned} \Delta S_{\text{comp}} &= S(n) - S(1) \\ &= k \ln(\exp(\varepsilon n / kT) - 1) - k \ln(\exp(\varepsilon / kT) - 1) + (E(n) - E(1))/T \end{aligned} \quad (5c)$$

While the computer moves down the energy ramp, it will on average pass heat  $-(E(n) - E(1))$  to the environment. As before this corresponds to a thermodynamic entropy increase in the environment of

$$\Delta S_{\text{env}} = -(E(n) - E(1))/T \quad (6c)$$

Thus the total thermodynamic entropy change is

$$\begin{aligned} \Delta S_{\text{tot}} &= \Delta S_{\text{comp}} + \Delta S_{\text{env}} \\ &= k \ln(\exp(\varepsilon n / kT) - 1) - k \ln(\exp(\varepsilon / kT) - 1) \\ &= k \ln\left(\frac{\exp(\varepsilon n / kT) - 1}{\exp(\varepsilon / kT) - 1}\right) \end{aligned} \quad (7c)$$

Rearranging (5c), we find that the change in free energy  $F=E-TS$  is

$$\Delta F_{\text{comp}} = -kT \ln\left(\frac{\exp(\varepsilon n / kT) - 1}{\exp(\varepsilon / kT) - 1}\right) \quad (8c)$$

We recover the same result from (3c) and the canonical expression  $F = -kT \ln Z$ .

For very small  $\varepsilon$ , such that  $\varepsilon \ll kT/n$ , the total thermodynamic entropy created reverts to the  $k \ln n$  of the undriven computer (5a), which is the minimum entropy creation:

$$\Delta S_{\text{tot}} = k \ln\left(\frac{\exp(\varepsilon n / kT) - 1}{\exp(\varepsilon / kT) - 1}\right) \approx k \ln\left(\frac{1 + \varepsilon n / kT - 1}{1 + \varepsilon / kT - 1}\right) = k \ln n$$

Correspondingly the free energy change  $\Delta F_{\text{comp}}$  reverts to  $-kT \ln n$  of (8a).

If  $\varepsilon$  is large so that  $\varepsilon \gg kT$ , the creation of thermodynamic entropy becomes linear in  $n$

$$\Delta S_{\text{tot}} = k \ln\left(\frac{\exp(\varepsilon n / kT) - 1}{\exp(\varepsilon / kT) - 1}\right) \approx k \ln\left(\frac{\exp(\varepsilon n / kT)}{\exp(\varepsilon / kT)}\right) = k \ln(\exp(\varepsilon(n-1)/kT)) = (n-1)\varepsilon/T$$

which grows with  $n$  much faster than the logarithm in  $k \ln n$ . Since large values of  $\epsilon$  would be needed to drive the system into its final stage with high probability, this method of assuring termination of the computation is thermodynamically costly.

### 3.6 The Energy Driven Brownian Computer with Trap

Finally, I will provide an abbreviated account of the case of a Brownian computer that is both driven by an energy gradient and brought to completion with an energy trap. Its Hamiltonian is a combination of the two earlier cases

$$H = \pi(\mathbf{p}) + \varphi(\lambda) \quad (1d)$$

where

$$\begin{aligned} \varphi(\lambda) &= -\epsilon\lambda && \text{for } 0 < \lambda < n \\ &= -\epsilon n - E_{\text{trap}} && \text{for } n < \lambda < n+1. \end{aligned}$$

It is illustrated in Figure 8.

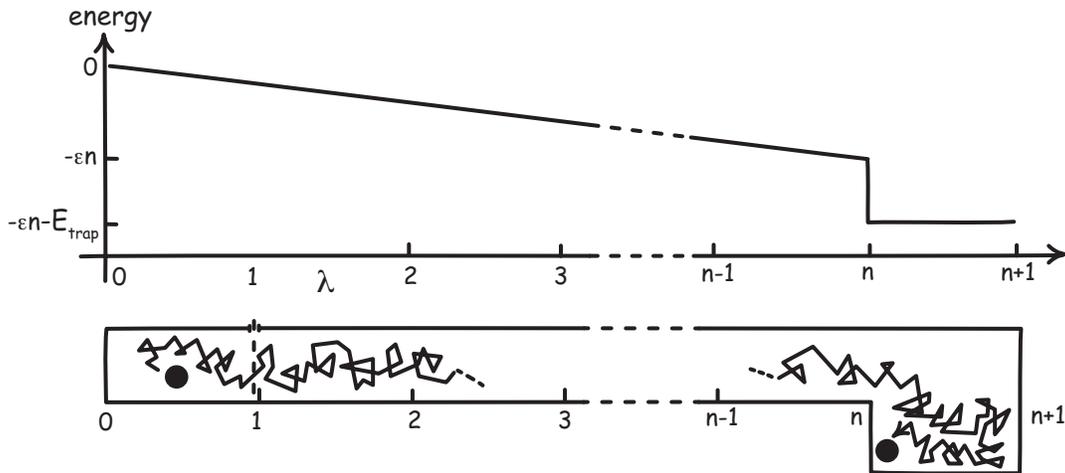


Figure 8. Energy Driven Brownian Computer with Energy Trap

Since this case incorporates both dissipative processes added in the last two cases, in operation it will create more thermodynamic entropy than any case seen so far, that is, in excess of  $k \ln n$ , so I will not compute the thermodynamic entropy created.

If  $P$  is the probability that the fully expanded system is in the trap, we can compute the odds ratio  $O_P = P/(1-P)$  by taking the ratio of the partition functions for the two regions of phase space:  $Z(n)$  for the first  $n$  stages and  $Z(\text{trap})$  for the final trap state  $n < \lambda < n+1$ . From (3c) and (3b) we have

$$Z(n) = M \cdot V \cdot (kT / \epsilon) \cdot (\exp(\epsilon n / kT) - 1)$$

$$Z(\text{trap}) = M \cdot V \cdot \exp((\epsilon n + E_{\text{trap}}) / kT)$$

We have for the odds ratio

$$O_p = \frac{P}{1-P} = \frac{Z(\text{trap})}{Z(n)} = \frac{\exp((\epsilon n + E_{\text{trap}}) / kT)}{(kT/\epsilon) \cdot (\exp(\epsilon n / kT) - 1)} = \frac{\epsilon}{kT} \cdot \frac{\exp(E_{\text{trap}} / kT)}{1 - \exp(-\epsilon n / kT)}$$

We can invert this last expression to yield

$$E_{\text{trap}} = kT \ln(kT/\epsilon) + kT \ln(1 - \exp(-\epsilon n / kT)) + kT \ln O_p \quad (13d)$$

It reverts to the corresponding expression (13b) for the undriven Brownian computer when we assume that  $\epsilon n / kT \ll 1$ , for then

$$kT \ln(kT/\epsilon) + kT \ln(1 - \exp(-\epsilon n / kT)) \approx kT \ln(kT/\epsilon) + kT \ln(\epsilon n / kT) = kT \ln n$$

If instead we assume more realistically that  $\epsilon n / kT \gg 1$ , so that  $\exp(-\epsilon n / kT) \approx 0$ , we recover

$$E_{\text{trap}} = kT \ln(kT/\epsilon) + kT \ln O_p = kT \ln(O_p kT/\epsilon)$$

This seems to be the result to which Bennett (1982, p. 921) refers when he writes:

However the final state occupation probability can be made arbitrarily large, independent of the number of steps in the computation, by dissipating a little extra energy during the final step, a “latching energy” of  $kT \ln(kT/\epsilon) + kT \ln(1/\eta)$  sufficing to raise the equilibrium final state occupation probability to  $1 - \eta$ .

The two results match up close enough if we set  $P = 1 - \eta$  and approximate  $O_p \approx 1/(1-P)$  when  $P$  is very close to unity. However the result does not conform quite as well with Bennett’s (1973, p. 531) remark that:

If all steps had equal dissipation,  $\epsilon < kT$ , the final state occupation probability would be only about  $\epsilon/kT$ . By dissipating an extra  $kT \ln(3 kT/\epsilon)$  of energy during the last step, this probability is increased to about 95%.

A final stage probability of  $P = 0.95$  corresponds to an odds ratio of  $O_p = 20$ , so that the extra energy dissipated should be  $kT \ln(20 kT/\epsilon)$ . Compatibility would be restored if we assume a missing “+” in Bennett’s formula, for  $\ln 20 = 3$ , so that

$$kT \ln(20 kT/\epsilon) = kT \ln 20 + kT \ln(kT/\epsilon) = kT(3 + \ln(kT/\epsilon))$$

## 4 Thermodynamic Reversibility is Mistakenly Attributed to Brownian Computers

The results of the last section can be summarized as follows. An  $n$  stage computation on a Brownian computer is a thermodynamically irreversible process that creates a minimum of  $k \ln n$  of thermodynamic entropy (see equation (7a)). Additional thermodynamic entropy of  $k \ln (1+O_p)$  is created to complete the computation by trapping the computer state in a final energy trap with probability odds  $O_p$  (see equation (7b)). If we accelerate the computation by adding an energy gradient of  $\epsilon$  per stage, we introduce further creation of thermodynamic entropy according to equation (7c). For a larger gradient, the thermodynamic entropy created grows linearly with the number of stages.

While it is thermodynamically irreversible, a Brownian computer is routinely misreported as operating thermodynamically reversibly. Bennett (1984) writes:

A Brownian computer is reversible in the same sense as a Carnot engine: Both mechanisms function in the presence of thermal noise, both achieve zero dissipation in the limit of zero speed, and both are, in accordance with thermodynamic convention, presumed to be absolutely stable against structural decay (e.g., thermal annealing of a piston into a more rounded shape), despite their being non-equilibrium configurations of matter.

This misreporting is especially awkward since the case of the Brownian computer is offered as the proof of a core doctrine in the recent thermodynamics of computation: that logically reversible operations can be computed by thermodynamically reversible processes. Bennett (1988, pp. 329-31) reports that the “proof of the thermodynamic reversibility of computation [of logically reversible operations]” arose through the investigation into the biochemistry of DNA and RNA that culminated in the notion of the Brownian computer. Bennett (2003, p. 531) reports that the objection against thermodynamically reversible computation of logically reversible operations “has largely been overcome by explicit models.” He then cites the non-thermodynamic, hard sphere model of Fredkin and Toffoli; and “at a per-step cost tending to zero in the limit of slow operation (so-called Brownian computers discussed at length in my review article; [(Bennett, 1982)])”

## 5. Thermodynamically Reversible Processes

Evidently, thermodynamically reversible processes can be hard to identify correctly. The above misidentification remains unchallenged in the literature. Hence it will be useful to review here just what constitutes thermodynamic reversibility and how it can be misidentified.

### 5.1 What They Are

The key notion in a thermodynamically reversible process is that all thermodynamic driving forces are in perfect balance. This traditional conception is found in the old text-books. Poynting and Thomson (1906, p. 264) give the “conditions for reversible working” that “indefinitely small changes in the external conditions shall reverse the order of change.” They list these conditions as: bodies exchanging heat “never differ sensibly” in temperature; and that the “pressure exerted by the working substance shall be sensibly equal to the load.” It follows that “exactly reversible processes are ideal, in that exact reversibility requires exact equilibrium with surroundings, that is, requires a stationary condition.” This means that nothing changes, so there is no process. They then offer the familiar escape:<sup>5</sup>

...we can approximate as closely as we like to the conditions of reversibility, by making the conditions as nearly as we like [to] those required, and lengthening out the time of change.

Planck (1903, §§71-73) gives an essentially similar account. He writes of pressures that differ “just a trifle,” “infinitesimal differences of temperature” and “infinitely slow” progress. The process consists of “a succession of states of equilibrium.” More fully:

If a process consists of a succession of states of equilibrium with the exception of very small changes, then evidently a suitable change, quite as small, is sufficient to reverse the process. This small change will vanish when we pass over to the limiting case of the infinitely slow process...

---

<sup>5</sup> It is quite delicate matter to explain the cogency of the notion of a thermodynamically reversible process when proper realization of the process entails that nothing changes, so no process occurs. For my attempt see Norton (forthcoming b).

We need only add to these classic accounts that generalized thermodynamic forces, such as those derived from (9) and which generalize the notion of pressure, must also be in balance.

When a Brownian particle is released into a liquid, its resulting exploration of the accessible volume is driven, thermodynamically speaking, by an unbalanced osmotic pressure, as Einstein argued in his celebrated analysis of 1905. Hence it is a thermodynamically irreversible expansion. Correspondingly, when a Brownian computer is set into its initial state and then allowed to explore the accessible computational space, the exploration is a thermodynamically irreversible process.

## **5.2 How We Might Misidentify Them**

There are many ways we may come to misidentify a thermodynamically irreversible process as thermodynamically reversible.

*Infinite slowness is not sufficient to identify thermodynamic reversibility.*

While thermodynamically reversible processes are infinitely slow, the converse need not hold. Sommerfeld (1962, p. 17) gives the simple example of an electrically charged capacitor that can be discharged arbitrarily slowly through an arbitrarily high resistance. While the process can be slowed indefinitely, it is a thermodynamically irreversible conversion of the electrical energy of the capacitor into heat. The driving voltage is not balanced by an opposing force. A simpler example is the venting of a gas at high pressure into a vacuum through a tiny pinhole. The process can be slowed arbitrarily, but it is not thermodynamically reversible since the gas pressure is unopposed.

*Reversibility of the microscopic or molecular dynamics not sufficient to assure thermodynamic reversibility.*

One cannot discern thermodynamic reversibility by affirming the reversibility of the individual processes that comprise the larger process at the microscopic or molecular level. They may be reversible, in the sense that they can go either way, while the overall process is itself thermodynamically irreversible. As a general matter, any thermodynamically irreversible process

may be reversed by a vastly improbable fluctuation. That possibility depends upon the microscopic reversibility of the underlying processes.<sup>6</sup>

A pertinent example is the thermodynamically irreversible expansion of a one molecule gas. Its momentary, microscopic behavior is reversible. To see this, consider a one molecule gas suddenly released into a large chamber filled with fixed, oddly shaped obstacles. If we simply attend to the molecule's motion over a brief period of time, while it undergoes one or two collisions, we would see mechanically reversible motions, as illustrated in Figure 9.

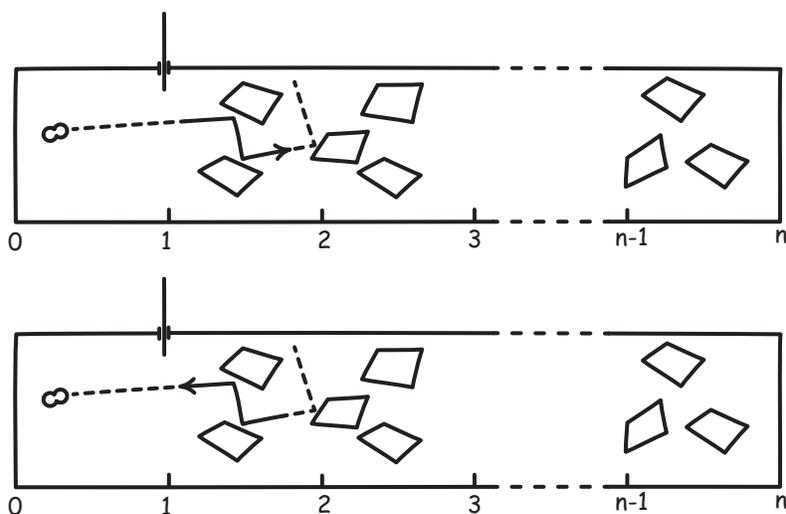


Figure 9. Microscopic Reversibility of a Thermodynamically Irreversible Expansion

However that would mislead us. We need to attend to the initial low entropy state of confinement of the one molecule gas; and its final high entropy state in which it explores the larger volume freely in order to recognize the thermodynamically irreversible character of the expansion.

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<sup>6</sup> For isothermal, isobaric chemical reactions, the relevant generalized force is the chemical potential  $\mu_A = (\partial G_A / \partial n_A)_{T,P}$ , where  $G = E + PV - TS$  is the Gibbs free energy of reagent A and  $n_A$  the number of moles of A. In dilute solutions,  $\mu_A = \mu_{A_0} + RT \ln [A]$  for R the ideal gas constant,  $\mu_{A_0}$  the chemical potential at reference conditions and  $[A]$  the molar concentration.

While each chemical reaction is reversible at the molecular level, the term  $RT \ln [A]$  contributes an entropic force, so that a chemical reaction will be thermodynamically irreversible if the concentrations of the reagents and products are not constantly adjusted to keep them at equilibrium levels.

Precisely the same must be said for both Brownian motion and a Brownian computer. They are both initialized in a state of low thermodynamic entropy; and then expand in a thermodynamically irreversible process to explore a larger space. At any moment, however, their motions will be mechanically reversible. We would be unable to tell whether we are observing their development forward in time or a movie of that development run in reverse. To separate the two, we would need to observe long enough to see whether the time evolution carries the system to explore the larger accessible space or whether it carries it back to its initial state of confinement.

Bennett (1982, p, 912) reports that a Brownian computer “is about as likely to proceed backward along the computational path, undoing the most recent transition, as to proceed forward.” Similarly Bennett and Landauer (1985, p. 54) report for the Brownian computer that “[i]t is nearly as likely to proceed backward along the computational path, undoing the most recent transition, as it is to proceed forward.”<sup>7</sup>

This sort of reversibility is insufficient to establish thermodynamic reversibility.

*Tracking internal energy instead of thermodynamic entropy is insufficient to identify thermodynamic reversibility.*

A thermodynamically reversible process is one in which the total thermodynamic entropy  $S_{\text{tot}} = S_{\text{sys}} + S_{\text{env}}$  remains constant, where  $S_{\text{sys}}$  is the thermodynamic entropy of the system and  $S_{\text{env}}$  that of the environment. Thermodynamically reversible processes must be identified by tracking this entropy. They cannot be identified by tracking internal energy changes.

What confounds matters is that we often track thermodynamic reversibility by means of quantities that carry the label “energy,” such as free energy  $F = E-TS$ . These energies are useable this way in so far as they are really measures of thermodynamic entropy adapted to special conditions. For example, Brownian computers implement isothermal processes while in thermal contact with an environment with which they exchange no work. Hence, if we have a thermodynamic process parameterized by  $\lambda$  so that  $d = d/d\lambda$ , then the constant entropy condition of thermodynamic reversibility for a computer “comp” in a thermal environment “env” is

$$0 = dS_{\text{tot}} = dS_{\text{comp}} + dS_{\text{env}} = dS_{\text{comp}} - dE_{\text{comp}}/T = - dF_{\text{comp}}/T.$$

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<sup>7</sup> I believe the “nearly” refers to the small external force they add corresponding to the energy ramp of Section 3.5 above.

It corresponds to constancy of the free energy  $F_{\text{comp}}$  of the computer.

Tracking internal energies gives the wrong result for Brownian computers. The thermodynamically irreversible  $n$  stage expansion of the Brownian computer is a constant energy process. The final energy trap could be replaced by a trap stage with a large volume  $V_{\text{trap}} = N_{\text{trap}}V$  of accessible configuration space. Then the final trapping can also be effected without any change of internal energy. The odds for the computer state being in the trap are  $O_P = P/(1-P) = N_{\text{trap}}/n$ . Using (7a), the total thermodynamic entropy created is

$$\begin{aligned}\Delta S_{\text{tot}} &= \Delta S_{\text{comp}} + \Delta S_{\text{env}} = k \ln (n + N_{\text{trap}}) \\ &= k \ln n + k \ln (1 + N_{\text{trap}}/n) = k \ln n + k \ln (1 + O_P)\end{aligned}$$

which agrees with the thermodynamic entropy creation of the energy trap (7b).

Bennett (1973, p. 531) introduced a small energy gradient in order to bring some “positive drift velocity” into Brownian computation. As we saw in Section 3.2 and equation (11), without it, no average speed can be assigned to Brownian motion. However it is also unnecessary for completion of the computational processes. They proceed as does any diffusion process, with progress increasing with the square root of time. That means the computation will take longer to complete. Since temporal efficiency is not the issue, there seems no point in including a superfluous source of thermodynamic irreversibility.

In assessing the thermodynamic efficiency of the Brownian computation of logically reversible functions, Bennett and Landauer do not track thermodynamic entropy. Rather they track the wrong quantity, energy. Bennett writes of energy “dissipated,” both as the energy  $\epsilon$  per step and in the trap energy or “latching” energy  $E_{\text{trap}}$ . See Bennett (1973, p. 531; 1982, p. 915, 921). Bennett and Landauer (1985, pp. 54-56) write of energy “expended” or “dissipated”:

A small force, provided externally, drives the computation forward. This force can again be as small as we wish, and there is no minimum amount of energy that must be expended in order to run a Brownian clockwork Turing machine.

and

The machine can be made to dissipate as small an amount of energy as the user wishes, simply by employing a force of the correct weakness.

If the energy  $\epsilon$  per stage is made arbitrarily small, the change of *internal* energy  $E$  of the Brownian computer will also become arbitrarily small. However it would be an elementary error

to confuse that with the operation of the computer becoming thermodynamically reversible, so that no net thermodynamic entropy is created; or to confuse it with the change in *free* energy  $F=E-TS$  becoming arbitrarily small. For one must also account for the “TS” term in free energy. For a Brownian computer driven by an energy gradient of  $\epsilon$  per stage, the free energy change is given by (8c). As we saw in Section 3.5, it reverts to the value  $-kT \ln n$  when  $\epsilon$  becomes arbitrarily small.

Finally, I will mention another confusion here, although it has only played an indirect role in the misidentification of Brownian computation. It is common to assign an additional thermodynamic entropy of  $k \ln 2$  to a binary memory device merely if we do not know the datum held in the device. As I have argued in Norton (2005, §3.2), this additional assignment is incompatible with standard thermodynamics. If one persists in using it, one will misidentify which are processes of constant thermodynamic entropy and thus which are thermodynamically reversible. Thus Bennett (1988; 2003, p. 502) describes erasure of a cell with “random data” as “thermodynamically reversible,” but one with “known data” as “thermodynamically irreversible.” Since this literature uses the same erasure process in both cases, it follows that whether a process is thermodynamically reversible depends on what you know. That is incompatible with thermodynamic reversibility as a (near) balance of physical forces. They will balance independently of what we know. To rescue these claims, we need to rebuild thermodynamics with new notions of entropy and reversibility. Ladyman et al. (2008) have tried to construct such an augmented thermodynamics. Norton (2011, §8) explains why I believe their efforts have failed.<sup>8</sup>

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<sup>8</sup> Bennett (1988, p. 329) writes:

When truly *random* data (e.g. a bit equally likely to be 0 or 1) is erased, the entropy increase of the surroundings is compensated by an entropy decrease of the data, so that the operation as a whole is thermodynamically reversible....When erasure is applied to such [nonrandom] data, the entropy increase of the environment is not compensated by an entropy decrease of the data, and the operation is thermodynamically irreversible.

To interpret these remarks, one needs to know that Bennett tacitly assumes an inefficient erasure procedure that also creates  $k \ln 2$  of thermodynamic entropy that is passed to the environment.

## 6 Relation to Landauer's Principle

Brownian computation is normally limited to logically reversible operations, so that the accessible phase space forms a linear channel. If it is applied to logically irreversible operations, the accessible phase space becomes branched, possibly exponentially so. This branching has been associated with Landauer's principle of the entropy cost of information erasure. I have argued elsewhere (Norton, 2005, 2011) that, even 50 years after its conception, the principle is at best a poorly supported speculation.<sup>9</sup> None of the attempts to demonstrate it have succeeded. Can Brownian computation finally provide the elusive justification? We shall see below that the principle gains no support from Brownian computation.

### 6.1 The Principle

Bennett (2003, p. 501) describes it as:

Landauer's principle, often regarded as the basic principle of the thermodynamics of information processing, holds that any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information-bearing degrees of freedom of the information-processing apparatus or its environment.

He then asserts a converse:

Conversely, it is generally accepted that any logically reversible transformation of information can in principle be accomplished by an appropriate physical mechanism operating in a thermodynamically reversible fashion.

### 6.2 Computing Logically Irreversible Operations

The simplest instance of logical irreversibility is erasure. An  $n$  stage erasure program applied to a single memory cell has two computational paths. One takes the cell, initially holding 0 to the end state, holding 0; the other takes a cell initially holding 1 to the end state 0. This

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<sup>9</sup> For other critiques of Landauer's principle, see Maroney (2005) and Hemmo and Shenker (2012, Ch. 11-12).

logical branching backwards from the 0 end state is implemented in a Brownian computer as backward branched channels in the accessible phase space, as shown on the top left in Figure 10.

While we may initialize the program to run on a cell holding, say, 0, when the computer state diffuses through the accessible phase space, it will also enter the other branch. This increases the accessible configuration space from  $nV$  to  $2nV$  and that will lead to a corresponding increase in thermodynamic entropy creation. The analyses of Section 3 still apply since they depend only on the accessible volume of phase space, not whether it has a linear or branched structure. For an undriven, trapped Brownian computer, replacing  $n$  with  $2n$  in (7b), we find that

$$\Delta S_{\text{tot}} = k \ln 2n + k \ln (1 + O_p) = k \ln 2 + k \ln n + k \ln (1 + O_p)$$

That is, there is an increase of thermodynamic entropy creation due to exploration of the additional branches of  $k \ln 2$ .

Figure 10 shows the more general case in which the program uses the same  $n$  stages sequentially to erase an  $N$  cell memory device, holding binary data.

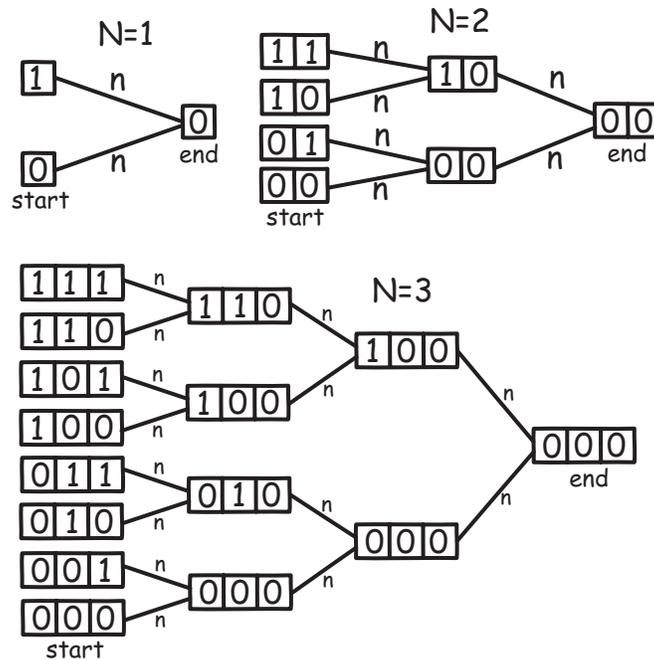


Figure 10. Accessible Configuration Space for an  $N$  Cell Erasure Program in Brownian Computation

The volume of configuration space accessible is

$$2nV + 4nV + 8nV + \dots + 2^N nV = nV(2^{N+1}-2)$$

Replacing  $n$  with  $n(2^{N+1}-2)$  in (7b), we find that

$$\Delta S_{\text{tot}} = k \ln n(2^{N+1}-2) + k \ln (1+O_p) = k \ln (2^{N+1}-2) + k \ln n + k \ln (1+O_p)$$

The increase in thermodynamic entropy creation is

$$k \ln (2^{N+1}-2) \tag{14}$$

### **6.3 Failure of the Connection to Landauer's Principle**

One might be tempted to see some sort of vindication of Landauer's principle in this entropy increase. It is not there.

The lesser problem is that expression (14) is the wrong formula. The Landauer limit for erasure of memory cells with binary data is  $k \ln 2$  per cell; that is  $Nk \ln 2$  for an  $N$  cell device. For large  $N$ , (14) approaches  $(N+1)k \ln 2$ .

The main problem is that nothing in the logical irreversibility of the erasure operation necessitates the creation of the entropy of (14). Rather, it is an awkward artifact of Brownian computation that it unnecessarily makes accessible volumes of phase spaces associated with unintended branches of the computation. In this regard it is akin to the category of failed proofs of Landauer's principle listed in (Norton, 2011, §3) as "proof by thermalization." Those proofs thermalize a memory device, thereby introducing an unnecessary thermodynamically irreversible expansion and then misreport the thermodynamic entropy created as a necessity of erasure.

The issue with Landauer's principle is to determine which operations can be carried out by thermodynamically reversible computations and which cannot; and to specify how much thermodynamic entropy the latter must create. Brownian computation is driven by thermodynamically irreversible processes. Hence it is the wrong instrument to use. That some Brownian computation creates some amount of thermodynamic entropy is no basis for determining that another device, operating in a thermodynamically reversible way, cannot do better.

Thermodynamic entropy is always created in Brownian computation. Its extent depends only on the volume of phase space into which the computation expands and not on whether the operation computed is logically reversible. Consider a logically reversible operation that chains  $(2^{N+1}-2)$  operations in series, such that each operation involves  $nV$  of configuration space. The operation is logically reversible but its computation will create exactly as much thermodynamic

entropy as the erasure of the N cell memory device above. What matters is not whether a logically reversible operation is computed, but whether the two computations are driven by the same expansion of phase space volume.

#### **6.4 Landauer's Principle as a Temporal Effect?**

Bennett's analysis differs from that just given, as it must. It cannot include the thermodynamic entropy (14), for his analysis neglects the entropic forces that create it. Instead, Bennett's concern is that exploration of the additional accessible phase will slow down the computation unacceptably. He writes (Bennett, 1982, p. 922)<sup>10</sup>

In a Brownian computer, a small amount of logical irreversibility can be tolerated ..., but a large amount will greatly retard the computation or cause it to fail completely, unless a finite driving force (approximately  $kT \ln 2$  per bit of information thrown away) is applied to combat the computer's tendency to drift backward into extraneous branches of the computation. Thus driven, the Brownian computer is no longer thermodynamically reversible, since its dissipation per step no longer approaches zero in the limit of zero speed.

That is, we must create more thermodynamic entropy to drive the computation forward to its end state and keep in out of the extraneous branches. Bennett (1973 pp. 531-32) gives the quantitative expression:

This in turn means (roughly speaking) that the dissipation per step must exceed  $kT \ln m$ , where  $m$  is the mean number of immediate predecessors 1) averaged over states near the intended path, or 2) averaged over all accessible states, whichever is greater. For a typical irreversible computer, which throws away one bit per logical operation,  $m$  is approximately two, and thus  $kT \ln 2$  is, as Landauer has argued ([Landauer 1961]), an approximate lower bound on the energy dissipation of such machines.

Bennett leaves unclear whether the "energy dissipation" indicated is derived from a computation not provided or is presumed on the prior authority of Landauer's principle. I will not pursue the

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<sup>10</sup> See also Bennett (1982, pp. 905-906, 923) for similar remarks.

question. Since this dissipation arises in addition to the entropy creation described in Section 6.1 above, it is at best only a part of the full account.

More generally, unless the branching structure introduces infinite phase volume, the extra dissipation is unnecessary and can provide no vindication of Landauer's principle. For Bennett's concern over the speed of computation is misplaced. It is standard in thermodynamics to allow processes unlimited but finite time for completion, so that they can approach thermodynamic reversibility arbitrarily closely. If one's interest is what is possible in principle by a thermodynamically reversible process, one should not create additional thermodynamic entropy merely to speed up the process. That will only confound the analysis.

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