ON BROWNIAN COMPUTATION

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Brownian computation, as presently characterized in the literature, is thermodynamically irreversible, the analog of an uncontrolled expansion of a one-molecule gas. I propose a modification to the protocol that brings the process as close to thermodynamic reversibility as the overcoming of thermal fluctuations allows.

1. Introduction

Brownian computation was introduced in papers by Bennett[1-2] and Bennett and Landauer[3] as a way of demonstrating that logically reversible mathematical operations can be computed by physical processes that are thermodynamically reversible, or nearly so. In a recent paper[4], I have argued that the computational processes they describe are *not* thermodynamically reversible and that their misidentification rested on a misapplication of thermal and statistical physics.

A question left open by my critique was whether there is some way to repair the computational protocol used in a Brownian computer so that its operation becomes thermodynamically reversible; or at least as close to thermodynamic reversibility for a single step process as the "no-go" result of Ref. 5 (Part II) allows. My purpose in this note is to describe such a repair.

It must be emphasized here that the repair is my proposal and that it conflicts with an assumption used in the Bennett and Landauer analysis. The proposed repair leads to a Brownian computer that will compute extremely slowly. Bennett (Ref. 2, pp. 905-906, 922-23; Ref. 1, pp. 531-32) has urged that if the Brownian computer's operation becomes too slow, a thermodynamically dissipative driving force must be introduced to drive it to completion. In the case of logically irreversible computation, the resulting dissipation is identified with the dissipation required by Landauer's Principle. My view is that processes approaching thermodynamically reversibility should be allowed arbitrarily long times for completion, as long as those times are finite.

2. A Brownian Computer with Energetically Suppressed Intermediate States

The repair is an adjustment to the "undriven Brownian computer with trap," described in Ref. 4 (Section 3.4). In this protocol, the computer starts in a stage $\lambda=0$ to $\lambda=1$, and

diffuses by a generalized Brownian motion to fill stages $\lambda=0$ to $\lambda=n$. It is then trapped probabilistically when it falls into a final stage $\lambda=n$ to $\lambda=n+1$ of low energy. The total thermodynamic entropy creation is given as

$$\Delta S_{tot} = k \ln n + k \ln (1 + O_P) \qquad (1)$$

where $O_P = P/(1-P)$ are the odds of successful trapping with probability P. Thermodynamically, the process is analogous to an n-fold, irreversible, isothermal expansion of a one-molecule gas, followed by the energy trapping. This irreversible expansion provides the largest contribution to the entropy creation of (1), the term $k \ln n$.

This expansion term can be made arbitrarily small by adjusting the energy of the intermediate stages in the configuration space. The final stage $\lambda=n$ to $\lambda=n+1$ is favored probabilistically by decreasing its energy to $-E_{trap}$. Using this same mechanism, the intermediate stages can be *dis*favored probabilistically by increasing their energy to E_{int}.[a] As the occupation of these stages becomes less probable, the expansion into them contributes less to thermodynamic entropy creation. A sufficiently large value for E_{int} can so reduce the occupation probability that the $k \ln n$ expansion term is replaced with one that is arbitrarily small.

There is a time penalty to be paid. A Brownian computer, initialized in stage $\lambda=0$ to $\lambda=1$, will have to surmount an energy mountain of height E_{int} before completing the computation, when it falls into the final stage energy trap. It will only be able to do that when a random fluctuation in its energy is large enough to let it climb the energy mountain. If the meandering of Brownian motion carries it back to the initial stage, there must be more fluctuations that allow it to climb the mountain again. Since an energy mountain of great height will be needed to suppress the expansion term, these fluctuations will be very infrequent.

The adjustments to the formulae governing the undriven Brownian computer with trap are as follows. The probability P that the computer is in the trap state is

$$P = \frac{\exp(E_{trap}/kT)}{1+(n-1)\exp(-E_{int}/kT)+\exp(E_{trap}/kT)} \approx \frac{\exp(E_{trap}/kT)}{1+\exp(E_{trap}/kT)} \qquad (2)$$

where the second approximation holds when E_{int} is made sufficiently large. The corresponding formula for the odds ratio is

$$O_P = \frac{P}{1-P} = \frac{\exp(E_{trap}/kT)}{1+(n-1)\exp(-E_{int}/kT)} \approx \exp(E_{trap}/kT) \qquad (3)$$

The partition integrals for the initial stage and all stages combined are

$$Z(1) = M.V \qquad Z(n+1) = M.V.(1 + (n-1)\exp(-E_{int}/kT) + \exp(E_{trap}/kT)) \qquad (4)$$

where, as before, M is the contribution of the momentum degrees of freedom and V the configuration space volume of each stage.

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[a] This increase is equivalent to merely decreasing the energies of both initial and final stages by E_{int}. It is easier to imagine how such a decrease might be implemented in a Brownian computer.

We can compute the total thermodynamic entropy change ΔS_{tot} of the computer and its thermal environment from the free energy change ΔF_{comp} of the computer alone using[b]

$$\Delta S_{tot} = -\Delta F_{comp}/T \tag{5}$$

From the expression $F = -kT \ln Z$ for free energy, we compute the change of free energy of the computer between the initial state, represented by $Z(1)$, and the final state, represented by $Z(n+1)$, as

$$\Delta S_{tot} = -\Delta F_{comp}/T = k \ln [1 + (n-1)\exp(-E_{int}/kT)] + k \ln (1+O_P) \tag{6}$$

The earlier expansion term $k \ln n$ has been replaced by

$$k \ln [1 + (n-1)\exp(-E_{int}/kT)] \tag{7}$$

When E_{int} is made sufficiently large, we have

$$\Delta S_{tot} \approx k \ln (1) + k \ln (1+O_P) = k \ln (1+O_P) \tag{8}$$

The expansion term has been reduced to insignificance, but not zero. That is the best we can do. The height of the energy mountain, E_{int}, must always be finite, else the computation cannot proceed. But a sufficiently large height for E_{int} can bring the expansion term (7) as close to zero as we like.

The total entropy (8) is the minimum entropy creation necessary to bring any single step thermodynamic process to completion, if thermal fluctuations are to be overcome, as is shown by the "no-go" result reported in Ref. 5 (Part II).[c] It is as close to thermodynamic reversibility as the no-go result allows.

3. Logically Irreversible Computation

The case dealt with so far is a Brownian computer implementing a logically reversible operation, so that the associated physical device is restricted to a one-dimensional channel in its configuration space. If a Brownian computer implements logically irreversible operations, such as erasure, that channel takes on a backward branched structure, as described in Ref. 4 (Section 6). We can extend the analysis of the last section to a device that implements a logically irreversible operation by considering a computer that erases—resets to 0—an N state binary memory device, as illustrated in Ref. 4 (Fig. 10). We will elevate the energies of the intermediate stages between the initial and final by an amount E_{int}. The effect once again is to suppress the expansion term corresponding to (7), but at the cost of a substantial slowing of the computation.

The associated entropy creation follows from slight adjustments to the formulae of the last section. The initial state partition integral will be

$$Z(init) = M.V \tag{9}$$

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[b] For a derivation, see Ref. 5 (Section 6).
[c] The corresponding equation of Ref. 5 (Section 9.2) asserts $\Delta S_{tot} = k \ln O_P$. The difference between this and Eq. (8) here is superficial. It arises from a slight difference in the definition of the entropy change computed. In this note, the entropy change is computed between the initial stage of the Brownian computer and an end state in which the computer is only probabilistically in the final trap stage. The formula of Ref. 5 assumes an extra step in which the system is subsequently isolated in its final stage, so that it is not just probabilistically, but assuredly in that final stage.

as before. The final state partition integral will now be

$$Z(fin) = M.V. (2^N + n.(2^{N+1}-2) \exp(-E_{int}/kT) + \exp(E_{trap}/kT)) \tag{10}$$

This is the partition integral that extends over the full tree accessible in configuration space. The first term, 2^N, reflects that the initial state of the full tree occupies 2^N separate volumes V of configuration space, one for each of the 2^N possible configurations of the memory device to be reset. The second term is drawn from the discussion of Ref. 4 (Section 6.2) and corresponds to the total volume of configuration space in all the intermediate stages of the tree, assuming that erasure of each bit requires n steps. The intermediate stages have their energy elevated in order to reduce the probability of occupation and the associated increase in thermodynamic entropy. The third term is associated with the end state energy trap.

As before, the contribution of the second expansion term of (10) can be made arbitrarily small by increasing the energy E_{int} sufficiently. Proceeding as above, what results is creation of thermodynamic entropy of

$$\Delta S_{tot} \approx k \ln (2^N) + k \ln (1+O_P) = Nk \ln 2 + k \ln (1+O_P) \tag{11}$$

The first term, $Nk \ln 2$, is an entropy creation of $k \ln 2$ for each of the N bits erased. This is the same quantity of entropy associated with erasure by Landauer's principle.

Elsewhere, I have argued that *none* of the demonstrations offered for Landauer's principle have succeeded.[d] Does this last computation finally provide a successful demonstration of Landauer's principle? It does not. All it shows is that a particular procedure for erasure *happens* to create $k \ln 2$ of thermodynamic entropy for each bit erased. What it does not show is that a procedure for erasure *has* to create this much entropy. The outcome (11) is quite compatible with Brownian computation merely being a thermodynamically uneconomical means of erasure, that is, one that creates thermodynamic entropy unnecessarily.

It becomes quite evident that this Brownian procedure creates entropy unnecessarily when we identify precisely how that unnecessary creation comes about. The two terms in equation (11) correspond to two entropy-creating processes. The second term, $k \ln (1+O_P)$, corresponds to the thermodynamic entropy that must be created to overcome thermal fluctuations and bring the process to completion with odds O_P. The first term arises from an uncontrolled expansion of the system from its confined initial state to fill the accessible phase space. This uncontrolled expansion is the standard illustration of an entropy creating, thermodynamically irreversible process. The creation of $Nk \ln 2$ of thermodynamic entropy is directly attributable to the protocol's inclusion of this thermodynamically irreversible step.

While this particular protocol for erasure happens to have this thermodynamically irreversible step, there is no demonstration that all possible variants of the Brownian protocol must do so; or that all possible erasure protocols, Brownian or not, must do so.

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[d] See Ref. 5 (Section 3.5) for a synopsis.

No necessity for the term is shown. Thus no demonstration of Landauer's principle is provided.

Thus, efforts to use the result (11) as a demonstration of Landauer's principle would fall into a familiar category of failed attempts. It is "erasure by unnecessarily dissipative thermalization," described in Ref. 5 (Section 3.5) An example of this category arises with memory devices consisting of chambers in each of which a partition traps a single molecule on one side. The erasure begins by removing the partitions, so that each one-molecule gas expands irreversibly to fill the chamber. This ill-chosen step creates k ln 2 of thermodynamic entropy unnecessarily and is the source of the k ln 2 of entropy that the demonstration attaches to erasure of each bit.

References

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