Einstein against black holes

Why Einstein thought what he thought

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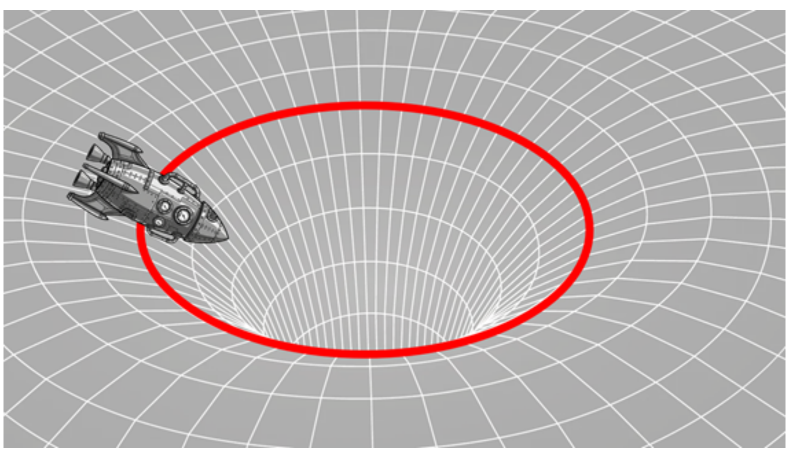
The event horizon of a black hole marks the point of no return. Once someone has passed it, even though spacetime in its vicinity is quite regular, they can no longer escape. Einstein and a roster of his leading contemporaries in mathematics and physics, however, in direct contradiction with our modern understanding, regarded spacetime there as breaking down, as 'singular', where terms in the equations disappear to zero or blow up to infinity. As University of Pittsburgh historian and philosopher of science John Norton explains, Einstein's concern about these terms now appears to us as a puzzling novice error. But a closer look at Einstein and the mathematical methods used reveals a different story.

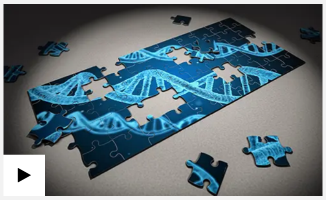
The moment

It is a moment in every introductory class on general relativity. The simplest, Schwarzschild black hole is introduced. Here is the singularity at its center. Here is the event horizon. Once it is passed, there is no escape. However, while passing it, an observer would not notice anything special in the vicinity of the event horizon. The spacetime geometry is quite regular there.



Then comes the warning. The commonly used mathematical description of spacetime has some anomalies at the event horizon, the "Schwarzschild radius." One term in the formula falls to zero and, worse, another diverges to infinity.

Do not be misled, there is no corresponding irregularity in the spacetime geometry. Its curvature stays finite. It is a novice mistake to think otherwise. These badly behaved terms just result from a peculiarity of the spacetime coordinate system used in the mathematics. If we choose another coordinate system, these pathologies go away. To think otherwise would be like imagining that there is a mystery at the Earth's north and south poles because our maps show all longitudes meet at them. But this is just an accident of how we chose to lay out lines of longitude on the Earth!



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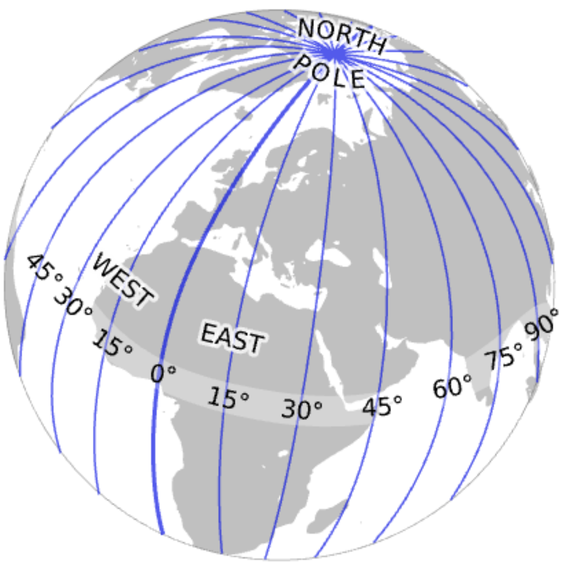
So, unlike the singularity now recognized at the center of a black hole, this divergence at the Schwarzschild radius, the event horizon, is not a real pathology of the spacetime, but rather a mathematical artifact that can be removed by changing the coordinate system used.



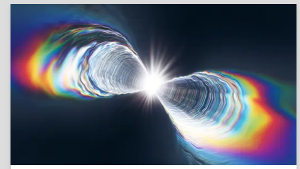
Lines of longitude converge at the North Pole

The puzzle

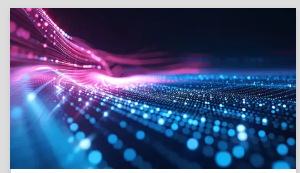
By modern lights, it is a novice mistake. Do only novices make it? Who did make it? It was the very physicist who discovered general relativity, Albert Einstein! He was quite sure that something goes terribly amiss at the Schwarzschild radius. When, in April 1922, the mathematician Jacques Hadamard asked Einstein what would happen if the radius were realized in a spacetime, Einstein replied in horror. It would be "an unimaginable misfortune [*malheur*] for theory…" He called it the "Hadamard catastrophe."

Einstein was quite determined in this assessment. George Lemaître showed in 1933 that the pathology could be eradicated merely by changing the coordinate system. Einstein was undeterred. In 1935, he coauthored a paper with Nathan Rosen in which they introduced the notion of the Einstein-Rosen bridge. We now draw it as the familiar funnel shape illustrated at the head of this article. They wrote in calamitous terms of

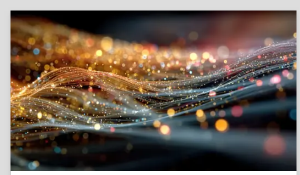
Spacetime isn't real in the way you think



To unify Relativity and Quantum Mechanics we must abandon materialism



The universe is not made of information



Quantum physics reveals there is no such thing as things

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the singularity they perceived to be at the Schwarzschild radius: “For a singularity brings so much arbitrariness into the theory that it actually nullifies its laws.” They were willing to alter Einstein’s celebrated gravitational field equations to escape the arbitrariness.

We have a serious puzzle to solve. Einstein, Hilbert, Klein. This is no collection of novices, but a roster of greatness in physics and mathematics. What were they thinking?

Perhaps Einstein’s assessment was an understandable mistake since, by his own admission, he had good physical instincts, but not so much in mathematics? No. David Hilbert, an early enthusiast for Einstein’s general theory of relativity, was a dominant figure in mathematics at the time. He gave an influential, mathematically precise exposition of it at the same time as Einstein was completing the theory in the mid-1910s. There, Hilbert endorsed Einstein’s claim of a supposed singularity at the Schwarzschild radius. The puzzle deepens.

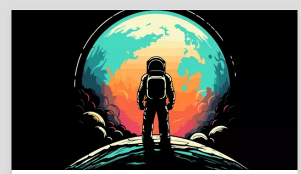
Again, might it be that neither Einstein nor Hilbert were good enough at geometry to see their mistake? Might a geometer have seen it? Hilbert’s colleague in Göttingen, Felix Klein, was a controlling figure in the development of geometry in the nineteenth and early twentieth centuries. In 1918, he alerted Einstein to a comparable problem in Einstein’s treatment of de Sitter’s spacetime. After Einstein responded, Klein relented and accepted Einstein’s appraisal.

We have a serious puzzle to solve. Einstein, Hilbert, Klein. This is no collection of novices, but a roster of greatness in physics and mathematics. What were they thinking?


What Einstein got wrong about time


The worst prediction in the history of science

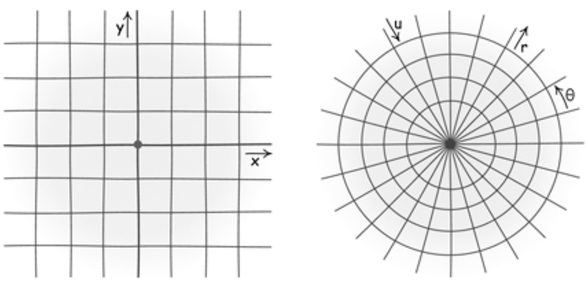

Einstein's biggest blunder with Claudia de Rham


Adventures in space

David Hilbert, Albert Einstein, Felix Klein

The mistake

To gain some perspective on the puzzle, here is a simplified version of the mistake attributed to Einstein. Consider the most familiar of geometries, that of a Euclidean space. We can map out its properties by using a Cartesian "*x*-*y*" coordinate system, as shown at the left. Or we might choose a radial "*r*-*θ*" coordinate system shown at right.



Cartesian and radial coordinate systems in a Euclidean space

The origin of the radial coordinate system seems to pick out something anomalous. It is a special point to which *all* angular coordinates θ are assigned. In analyzing the Kepler problem in astronomy, it is convenient to use a new coordinate $u = 1/r$. If we use this new coordinate, the origin $r = 0$ now corresponds to an infinity: $u = 1/0 = \infty$.

Infinity or not, the essential point is that the geometry of this space is everywhere regular. All these oddities simply arise from the fact that the radial coordinate system is poorly adapted to the one point that was arbitrarily selected to be its origin. It is a novice mistake to fail to see this.

Einstein, it is supposed, made just this mistake when he analyzed the Schwarzschild spacetime. Here is the key formula for the spacetime line element as it appears in the publication of lectures he gave in Princeton in May 1921:

$$ds^2 = \left(1 - \frac{A}{r}\right)dl^2 - \frac{dr^2}{1 - \frac{A}{r}} - r^2 d\phi^2 . \quad (109b)$$

Einstein’s formula for the Schwarzschild line element

The Schwarzschild radius is located at the radial coordinate $r = A$. At this value, the first term in this expression drops to zero, and the second term diverges to infinity. This divergence is interpreted as there being an infinite spatial distance, *ds,* between two points in space in the vicinity of the Schwarzschild radius that are separated by a radial coordinate difference *dr*. This is the pathology that so bothered Einstein.

Geometry versus analysis

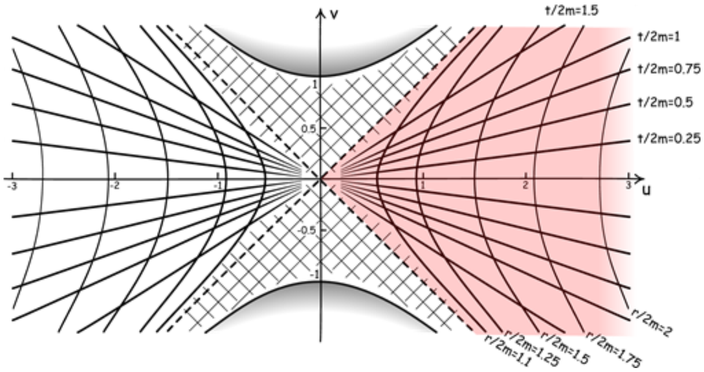
Why would Einstein and Hilbert accept that this divergence is a true spacetime pathology, a singularity? The answer lies in a difference in mathematical traditions whose existence has been largely forgotten. There are two elements in the analysis. One is the geometry intrinsic to a space or a spacetime. The other is the analytic expressions—the formulae we write down—that describe the space or spacetime.

These two elements almost always work well together. However, they may indicate different courses when problems arise. Then the decision is whether to follow the direction indicated by the geometry or that indicated by the analysis.

The divergence in Einstein’s formula above is such a case. If we allow geometric ideas to guide us, we conclude that the Schwarzschild spacetime remains fully regular at the Schwarzschild radius and that the divergence in Einstein’s formula arises from his coordinate system being ill-adapted to the spacetime geometry.

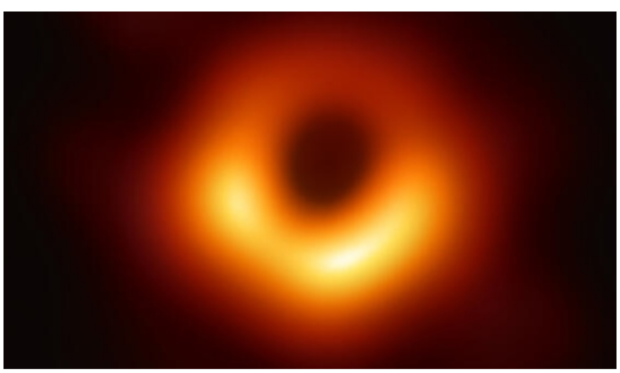
Geometrical intuitions guide us to adopt this fully extended Schwarzschild spacetime as the right course to take. That it is so is not a matter of some pre-ordained, a priori knowledge. It is empirical. It depicts better the way spacetime has turned out to be.

This is seen in the figure that depicts the fully-extended Schwarzschild spacetime. It is based on a transformation to a new Kruskal coordinate system with coordinates *u,v* that covers the entirety of the Schwarzschild spacetime. The true curvature singularities in spacetime, past and future, are shown as the hyperbolas at the top and bottom of the figure. These curvature singularities are located at the center of the black hole at "*r*=0" in Einstein's coordinate system. The part of this coordinate system *t*, *r* used by Einstein to describe the spacetime outside the Schwarszschild radius covers only a wedge of the spacetime, shown as the shaded region. The surfaces of constant *t* (the bold, straight lines on the figure) converge to a single event, much like the origin of the radial coordinate system in a Euclidean space.



Fully extended Schwarzschild spacetime

Geometrical intuitions guide us to adopt this fully extended Schwarzschild spacetime as the right course to take. That it is so is not a matter of some pre-ordained, *a priori* knowledge. It is empirical. It depicts better the way spacetime has turned out to be. Where Einstein's understanding would preclude the reality of black holes, LIGO's detection of gravitational waves has given us evidence of black hole coalescences. The most vivid affirmation of the geometric approach comes in the memorable images produced by the Event Horizon Telescope of, as the name suggests, the event horizon of a black hole.



The Event Horizon Telescope image of the black hole in Messier 87

Einstein's alternative

Einstein, of course, knew nothing of these results. They were far into the future. For him, the decision on how to address the divergence was a matter of physics. The key physical question is whether Einstein would have been prepared to accept what is shown in the figure of the fully extended Schwarzschild spacetime: that in this picture, the t surfaces of his coordinate system intersect at the r value of the Schwarzschild radius. This intersection would mean that, at this radius, all values of t are assigned to the same event, just as with q in the radial coordinate system example.

It will be puzzling to modern readers that Einstein could so easily dismiss the coordinate transformation that dissolves the divergence. That he could and should dismiss it followed from his privileging of analytic expressions. For Einstein, the content of general relativity lay in the analytic expressions that describe the spacetime. More precisely, each spacetime was represented by a set of formulae that consists of the formulae in one coordinate system together with the formulae that result from it for all admissible coordinate transformations.

The stumbling block is the determination of just which coordinate transformations are admissible. A modern geometer would want to include the transformation from Einstein's r, t coordinate system to Kruskal's u, v coordinate system. Einstein, however, required that the admissible coordinate transformations must be "one-to-one." That is, the transformation must map each pair r, t to exactly one pair u, v; and the reverse transformation must map each pair u, v to exactly one pair r, t. This last condition fails for the transformation to Kruskal coordinates. To see it, look at the event labeled by Kruskal coordinates u=0 and v=0 in the figure above. That event is assigned the coordinate r=2m in Einstein's coordinate system and *all values* of the coordinate t from $-\infty$ to $+\infty$. The transformation that would eradicate the divergence is blocked by the demand that it be one-to-one.

We might still want to discount Einstein's attachment to one-to-one coordinate transformations as a novice, mathematical error. We cannot do so. David Hilbert, the authority in mathematics of Einstein's time, endorsed this restriction in his accounts of general relativity.

That Einstein privileged his analytic formulae and discounted geometric pictures is apparent from an examination of his writing in general relativity. We now understand the vanishing of Riemann's "curvature tensor" as the condition for geometric flatness of a spacetime. Einstein systematically avoided such geometrical construals. For him, its vanishing was merely flagging an algebraic property of his formulae: the existence of a spacetime coordinate transformation in which the line element of a spacetime would adopt the constant coefficient values characteristic of special relativity.

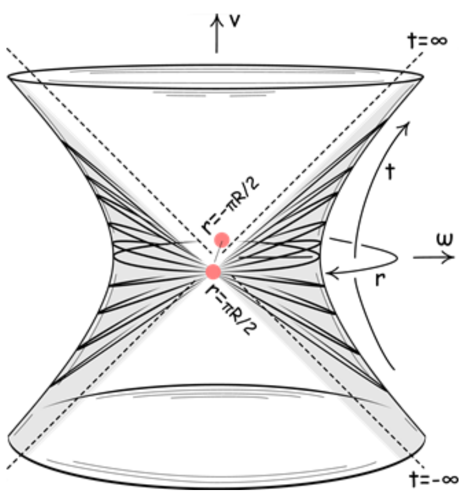
Direct evidence of Einstein's dismissive attitude to these geometric pictures has been found by Dennis Lehmkuhl. In correspondence of April 24, 1926, with the philosopher of science, Hans Reichenbach, Einstein dismissed geometrization as “a kind of novice aid [*Eselsbrücke* = ‘donkey bridge’].”

Einstein’s physical reasons

Einstein allowed his understanding of the physical basis of a spacetime to overrule what the simple geometric picture might indicate. This course becomes quite apparent in his treatment, later in the 1910s, of the spacetime introduced by Willem de Sitter in 1917, as a solution to Einstein’s gravitational field equations, augmented by Einstein’s cosmological constant.

The geometry of the de Sitter spacetime is now readily recognized as belonging to a four-dimensional hyperboloid embedded in a five-dimensional Minkowski spacetime. Five dimensions may make this sound complicated, but this prescription masks the great simplicity of the spacetime. It is analogous to a simple device for generating a non-Euclidean, two-dimensional spherical space. In such a space, angles of triangles sum to greater than two right angles and all straight lines end up curving back onto themselves. The strangeness of these properties disappears when we see that it is just the geometry of great circles on two-dimensional spheres in ordinary three-dimensional space.

The de Sitter spacetime is everywhere regular and has no singularities. Two of its dimensions are represented as the geometry induced on the surface of the hyperboloid in the figure.



De Sitter Spacetime

The great simplicity of this spacetime was quite troublesome for Einstein for physical reasons. First, if one located free motions in it in a natural way, they corresponded to a motion of uniform expansion or contraction. (Hence the de Sitter spacetime has a special place in present-day expanding universe cosmology.) The second problem was that it was a matter-free, since then the cosmological constant was not interpreted as a matter term.

In 1917, Einstein thought that the universe was static and, more deeply, he resolutely believed on epistemological grounds that a matter-free spacetime was impossible. This last condition became his “Mach’s Principle.” These beliefs on the physics do not conform with a spacetime realized as the full hyperboloid of de Sitter spacetime.

Once again, a geometer would have to say that there is no singularity in the spacetime. The appearance of one is merely an artifact of a poorly adapted coordinate system.

Einstein needed an escape. It lay in writing a line element for the de Sitter spacetime in which the spacetime appeared static, in the sense that its spacetime geometry was independent of the time coordinate t he chose. More significantly, its ordinary three-dimensional space was surrounded by a singularity that arose in the line element. This singularity signaled to Einstein where the mass-free line element no longer applied, and the theory broke down. The singularity was simply a place-holder for the matter of an as-yet undiscovered matter theory that would supply the matter needed to sustain the spacetime. It came to be known as the “mass horizon.”

We can now see that both these properties that so appealed to Einstein were artifacts of the coordinate system he chose. That coordinate system is illustrated in the figure as the r, t coordinates in the greyed-out wedges. The mass horizon singularity arises when the surfaces of constant t intersect at $r = R\pi/2$ and $r = -R\pi/2$.

Once again, a geometer would have to say that there is no singularity in the spacetime. The appearance of one is merely an artifact of a poorly adapted coordinate system. That “a geometer” would say this is no idle speculation. It is precisely what the great geometer, Felix Klein, pointed out to Einstein in letters of May 31 and June 16, 1918. He told Einstein that the mass horizon singularity “can be simply transformed away.” In his published writing, Klein even drew an analogy to the apparent singularity at the origin of a polar coordinate system.

The greatest living geometer had just accused Einstein of an elementary error in geometry. In his reply of June 20, Einstein was apparently untroubled. He quietly and patiently explained that the transformation Klein envisaged requires two distinct surfaces of constant t to intersect, whereas, in Einstein’s physical understanding, they did not intersect. That is, Klein’s transformation violated the one-to-one condition so painstakingly laid out by Klein’s Göttingen colleague, Hilbert.

It was a moment in history. The great physicist Einstein and the great geometer Klein facing off. Who would blink first? It was Klein. In subsequent publications, he reaffirmed the simple geometry of the full de Sitter hyperboloid, but accepted that Einstein had shown that the full hyperboloid was inadmissible as a real spacetime for physical reasons.

Two mathematical traditions

One would be mistaken to imagine that Einstein's dismissal of geometry was a personal aberration. Rather, it reflected a division in two mathematical traditions that had arisen in the nineteenth century. One was Carl Friedrich Gauss's theory of the geometry of curved surfaces. The second was Elwin Bruno Christoffel's analysis of the invariants of quadratic differential forms. Einstein worked in this latter tradition. His expression for the line element of Schwarzschild spacetime is just a quadratic differential form.

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Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades.

(Von Herrn *E. B. Christoffel* in Zürich.)

1.

Führt man in den Differentialausdruck

$$F = \Sigma\omega_{ik}\partial x_i \partial x_k; \quad i, k = 1, 2, \ldots n,$$

in welchem die Coefficienten ω beliebige Functionen der von einander unabhängigen Variabeln $x_1, x_2, \ldots x_n$ sind, statt der letztern ein System von einander unabhängiger Functionen der neuen Variabeln $x'_1, x'_2, \ldots x'_n$ ein, so geht F in einen neuen Differentialausdruck

$$F' = \Sigma\omega'_{ik}\partial x'_i \partial x'_k$$

über, welcher dem ursprünglichen vermöge der ausgeführten Substitution gleich ist.

Sind umgekehrt die beiden Differentialausdrücke F und F' gegeben, so kann man sich die Frage vorlegen, welche Bedingungen erforderlich sind, damit der eine in den andern transformirt werden könne, und falls dies möglich ist, welche Substitutionen die verlangte Transformation leisten.

Wir bezeichnen für diese Untersuchung die Substitutionsdeterminante und die Determinanten von F und F' wie folgt:

$$\Sigma \pm \frac{\partial x_1}{\partial x'_1} \frac{\partial x_2}{\partial x'_2} \cdots \frac{\partial x_n}{\partial x'_n} = r,$$
$$\Sigma \pm \omega_{11} \quad \omega_{22} \cdots \omega_{nn} = E,$$
$$\Sigma \pm \omega'_{11} \quad \omega'_{22} \cdots \omega'_{nn} = E'.$$

Dann findet, wie aus der algebraischen Invariantentheorie bekannt ist, eine und auch nur eine Transformationsbedingung

$$E' = r^2 E$$

statt, und diese würde auch für den vorliegenden Fall ausreichen, wenn die Coefficienten ω von F und die Elemente von r constant wären. Im gegenwärtigen Falle müssen jedoch noch andere Bedingungen erfüllt sein, da nicht jedes System linearer und homogener Functionen von $\partial x'_1, \partial x'_2, \ldots \partial x'_n$, welches an Stelle von $\partial x_1, \partial x_2, \ldots \partial x_n$ gesetzt F in F' überführt, die ge-

First page of Christoffel's 1869 "On the Transformation of Homogeneous Differential Expressions of the Second Degree"

This latter mathematical tradition could be applied to Gauss's theory and thus could be used as a geometric theory. However, its practitioners were careful to insist upon its independence from geometry, which was just one application of what was conceived as a more general tool, a general calculus.

What now looks to us as a novice error only does so because of the hard work of the generations after Einstein.

In a legendary moment in the history of general relativity, in 1912, an overwhelmed Einstein asked his mathematician friend, Marcel Grossmann, for mathematical help. Grossmann went to the library and there found Ricci and Levi-Civita's 1900 review of the "Methods of the absolute differential calculus and its applications." This was the mathematics tailor-made for Einstein's new theory of gravity.

Ricci and Levi-Civita were working in the latter tradition of the invariants of quadratic differential forms. Grossmann sympathized with the desire of that tradition to keep geometry as just one application of the calculus. When, in 1913, Einstein and Grossmann published the first sketch of what would become the general theory of relativity, Grossmann wrote the mathematical part of the work. There, in introducing the new mathematical methods, Grossmann expressed strong hesitations about the value of geometric thinking, just like those held by Einstein. Grossmann wrote:

> … I have deliberately set aside geometrical aids, since, in my opinion, they contribute little to the intuitive understanding [*Veranschaulichung*] of the formation of concepts of the vector analysis

What are we, then, to think of this episode? What of the judgments of Einstein, Grossmann, Hilbert, Klein, and a generation of accomplished physicists who agreed with them? It is the old saw: "hindsight is 20-20." What now looks to us as a novice error only does so because of the hard work of the generations after Einstein. It took almost half a century to see further than Einstein and to understand that there was a better way to treat the divergences he found in his theories of space and time.

Further reading: John D. Norton, "Einstein Against Singularities: Analysis versus Geometry." *Philosophy of Physics* 2(1)(2024): 13, 1–73.

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