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How Einstein found his field equations: 1912–1915

1. INTRODUCTION

BY THE MIDDLE of 1913, after less than nine months of collaboration with his mathematician friend Marcel Grossmann, Einstein had discovered virtually all the essential features of his general theory of relativity. For they had succeeded in constructing a gravitation theory in which the laws of nature could be written in a generally covariant form, that is in a form which remained unchanged under all coordinate transformations.[1] But, as Einstein confided to Lorentz, their new theory, the so-called *Entwurf* theory, was marred by an "ugly dark spot."[2] Its gravitational field equations, its most fundamental equations, were not generally covariant. It was not until November 1915 that Einstein could present the now familiar generally covariant field equations of the theory to the Prussian Academy of Science. In all, he had spent some three troubled years wrestling with the problem of these field equations.

Some of the highlights of this struggle are now well known. In the *Entwurf* paper, Einstein and Grossmann had come within a hair's breadth of the generally covariant field equations of the final theory.

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1. Their outline of the theory appeared first as a separatum, A. Einstein and M. Grossmann, *Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation* (Leipzig 1913), and then in the journal *Zeitschrift für Mathematik und Physik, 62* (1913), 225-261, with an addendum not in the separatum. Page references will be to the later version.

2. A. Einstein to H. A. Lorentz, 16 Aug 1913, EA 16 434.

They had considered field equations based on the Ricci tensor—a choice virtually forced on them by the mathematical requirements of general covariance. But they discarded these equations on the ground that they failed to yield the correct Newtonian limit. Shortly after, Einstein came to believe that he had found two proofs for the physical unacceptability of all generally covariant field equations. The more notorious of these was his so-called "hole" argument.

Unfortunately it has become common to dismiss these *crucial* turning points in Einstein's work in terms of barely excusable errors, even as simple mathematical slips by Grossmann or Einstein. The argument runs as follows. Generally covariant equations hold by definition in all coordinate systems, whereas the equations of Newtonian gravitation theory do not. So, in the process of recovering Newtonian theory as a limiting case from a generally covariant theory, it is necessary to restrict the set of coordinate systems under consideration. This is usually achieved through the explicit stipulation of a number of additional relations—called "coordinate conditions"—that must also be satisfied by the final solution. But—the argument continues—Einstein and Grossmann were ignorant of their freedom to apply such coordinate conditions and so failed to recover the correct Newtonian limit. Moreover, in working out his "hole" argument Einstein is supposed not to have recognized the elementary fact that a given physical instance of a gravitational field will be represented by different mathematical functions in different coordinate systems.

My purpose in this paper is twofold. First I will seek to establish that Einstein was fully aware of his freedom to apply coordinate conditions to generally covariant field equations and knew how the process could help recover a Newtonian form from such equations. The evidence for this is contained primarily in one of Einstein's notebooks from this period and is, I think, irrefutable. Second, I will develop a more satisfactory account of Einstein's struggles with his field equations in these three troubled years. In particular, I will be concerned to show that Einstein's difficulties were based on nontrivial misconceptions and that the path he followed was a thoroughly reasonable one. Stachel was the first to try to approach the problem in this way.[3] He has concluded that:

3. J. Stachel, "Einstein's search for general covariance, 1912–1915," paper read at the Ninth International Conference on General Relativity and Gravitation, Jena, 1980. Aspects of Stachel's second thesis had been anticipated in J. Earman and C. Glymour, "Lost in the tensors: Einstein's struggles with covariance principles 1912–1916," *Studies in history and philosophy of science*, 9 (1978), 251–278.

- Einstein's understanding of the form of static gravitational fields in 1913 was inconsistent with his final general theory of relativity and, moreover, with the Ricci tensor as a gravitation tensor. This alone could account for Einstein and Grossman's rejection of the Ricci tensor.

- Einstein's "hole" argument admits a reading in which it focuses on a serious physical problem in the relationship between the spacetime manifold and the gravitational field. This reading alone is consistent with Einstein's later resolution of the argument and, in this form, can be seen to contribute decisively to Einstein's understanding of spacetime in his new theory.

These two points are essential to the account I offer here of Einstein's work on his field equations in the three years ending in November 1915. In outline, my account runs as follows.

The question whether Einstein was aware of his freedom to apply coordinate conditions to generally covariant field equations at the time of the *Entwurf* paper will be settled by examination of the contents of one of his notebooks from this period of his work in Zürich. It will be clear that Einstein was fully aware of this freedom and even knew of two different coordinate conditions that could be used to reduce the Ricci tensor to a Newtonian form. But, I shall argue, Einstein was not prepared to accept either condition because of a number of related misconceptions.

At the heart of these misconceptions lay the problem of the circumstances under which the ten gravitational potentials of the new theory would reduce to a more manageable single potential. On the basis of his earlier work on gravitation and the principle of equivalence, Einstein believed that there was such a reduction in the case of static fields. He chose the simplest and most natural weak-field equations and again found that they led to a similar reduction in the number of gravitational potentials. These and other signposts all pointed in the same direction. But, unfortunately for Einstein, it was the wrong direction. Both his assumptions about static fields and about the weak-field equations were inconsistent with his final theory. In addition he had one final and puzzling misconception about the form of these weak-field equations in rotating coordinate systems. Together, these were sufficient to thwart Einstein's attempts to construct acceptable generally covariant field equations from the Ricci tensor.

The suspension of the requirement of general covariance was soon to follow. Through these same misconceptions, Einstein convinced himself that if derivatives up to the second order only were considered, the conservation of energy and momentum led to a unique set of field equations which were not generally covariant. This formed the substance of his derivation of the *Entwurf* field equations and precluded any further search for generally covariant field equations.

Now convinced of the fruitlessness of this search, Einstein developed general arguments against the physical acceptability of all generally covariant field equations. The first of these was based on the impossibility of constructing a generally covariant conservation law in which the energy-momentum of the gravitational field and of other matter would each be represented by a generally covariant tensor. This argument was soon eclipsed by what appeared to be a far stronger one, the notorious "hole" argument. In it, he purported to show that generally covariant field equations could not uniquely determine the field generated by certain simple distributions of source masses, in contradiction with the requirement of physical causality.

Contrary to the usual account, the "hole" argument was not based on the naive misunderstanding that a given gravitational field is somehow physically changed by the transition to a new coordinate system simply because the mathematical functions that represent it have changed. But it still failed to establish the untenability of generally covariant field equations. Rather, the argument amounted to a demonstration that generally covariant field equations cannot uniquely determine the field as long as the point events of the spacetime manifold are incorrectly thought of as individuated independently of the field itself. More figuratively, it showed that if one could somehow take away the field, one would not be left with a bare spacetime manifold replete with individual points. Nothing, not even this, would remain. Einstein did not interpret his "hole" argument in this way until his return to general covariance in November 1915.

Einstein's move from Zürich to Berlin in April 1914 ended his collaboration with Grossmann. But before this, Grossmann was still able to help Einstein with many of the preliminaries of the task of refining the mathematical formulation of the *Entwurf* theory and resolving the question of the theory's exact relation to the generalized principle of relativity. By means of variational techniques, Einstein developed a general way of formulating those field equations, which had exactly the maximum covariance permitted by his "hole" argument. He believed that this analysis led uniquely to his original *Entwurf* field equations, without any significant use of empirical knowledge of gravitation. Einstein was especially pleased with this outcome because it clearly demonstrated that the foundations of his new theory lay in covariance considerations. But he was unaware that his analysis by no means led uniquely to his *Entwurf* field equations. In the last step of his derivation he had made a mistake, for which I have been unable to find any explanation.

However, it has not generally been noted that Einstein's work on this question was not entirely in vain. He was able to use the mathematical machinery that he developed for it virtually unchanged in 1916 in his analysis of his final generally covariant field equations of

November 1915. In particular, he developed a device for yielding four "adapted" coordinate conditions. These conditions had to be satisfied if the *Entwurf* field equations were to hold in a given coordinate system. With his final generally covariant field equations of November 1915, this device yielded an important set of identities, now known as the contracted Bianchi identities, from which the conservation laws could be derived.

Einstein's return to the search for generally covariant field equations towards the end of 1915 came after a period of growing dissatisfaction with his *Entwurf* theory. The theory did not account for the known anomaly in the motion of Mercury and he found, contrary to his earlier belief, that it was not covariant under transformations to rotating coordinate systems, which he felt was required by the generalized principle of relativity. His return to this search was precipitated by his discovery of the mistake in the final step of his derivation of the *Entwurf* field equations in 1914.

But Einstein could not yet proceed directly to his final generally covariant field equations, for he was still bedevilled by the same virtually untouched misconceptions about static fields and about the Newtonian limit as he had had three years earlier. The unravelling of these misconceptions can be traced in a dramatic series of weekly communications from Einstein to the Prussian Academy, beginning on November 4, 1915. Hitherto, the story behind Einstein's apparently erratic turns in this final month has remained untold and was, perhaps, untellable, without the clues from the Zürich notebook.

Einstein's first step was to return to a set of almost generally covariant field equations that he had considered with Grossmann three years earlier. Then, however, he had rejected them because he believed that they failed to yield the required weak-field equations in a rotating coordinate system. A week later he showed how the adoption of the hypothesis that all matter is electromagnetic in nature enabled a modification of these equations, which at last was generally covariant and which also satisfied the restrictive requirements of his enduring misconceptions about static fields and the weak-field equations. But these were still not the field equations of his final theory. Einstein was freed from the misconceptions that separated him from these equations through his successful calculation of the orbit of Mercury, which he reported the following week. There he was confronted with a static field in which the ten gravitational potentials did not reduce to a single potential in the way he had expected for so long.

This freed him to entertain a wide range of generally covariant field equations and the following week, with a weary tone of finality, he reported the generally covariant field equations of the final theory to the Prussian Academy. I argue that Hilbert's simultaneous discovery of these equations played little if any role in Einstein's final solution, despite the intense correspondence between them at this time. The delay in Einstein's discovery can be explained entirely in terms of the difficulties outlined here and a natural pathway to the final equations can be reconstructed for that final week. In addition, there is evidence that Einstein was unaware of the exact nature of Hilbert's discovery for several months.

2. PRELUDE: FROM 1905 TO 1912

In June 1905, while still a patent examiner in Bern, Einstein submitted his famous work on the electrodynamics of moving bodies to the *Annalen der Physik*. This work contained his special theory of relativity, in which he asserted the equivalence of all inertial frames of reference as a fundamental postulate of physics. The question which then naturally arose, he recalled later, was whether it was possible to extend this principle of relativity to the more general case of frames of reference in arbitrary states of motion.[4] But he could find no workable basis for such an extension, until he tried to incorporate gravitation into his new special theory of relativity for a review article in 1907.[5] The difficulties of this task led him to a new principle, later to be called the "principle of equivalence."

On the basis of the fact that all bodies fall alike in a gravitational field, Einstein postulated the complete physical equivalence of a homogeneous gravitational field and a uniform acceleration of the frame of reference. This, Einstein noted in his review article, extended the principle of relativity to the case of uniform acceleration. It also foreshadowed the problem whose complete solution would lead him to his general theory of relativity: the construction of a relativistically acceptable theory of gravitation, based on the principle of equivalence.

Einstein did not publish any further on this question until 1911 and 1912, the years of his stay in Prague. Then he developed his speculations of 1907 into a substantial and innovative theory of static gravitational fields.[6] His strategy was simple. He would consider a frame of reference in uniform acceleration. According to his principle of

4. Einstein, "Notes on the origin of the general theory of relativity" (1933), in Einstein, *Ideas and opinions* (London 1973), 285–290, on 286–287.

5. Einstein, "Über das Relativitätsprinzip und die ausdemselben gezogenen Folgerungen," *Jahrbuch der Radioaktivität und Elektronik, 4* (1907), 411–462, and *5* (1908), 98–99.

6. Einstein, "Über den Einfluss der Schwerkraft auf die Ausbreitung des lichtes," *AP, 35* (1911), 898–908; "Lichtgeschwindigkeit und Statik des Gravitationsfeldes," *AP, 38* (1912), 355–369; and "Zur Theorie des statischen Gravitationsfeldes," *AP, 38* (1912), 443–458.

equivalence, the acceleration yielded a homogeneous gravitational field whose effect on a given phenomenon could be readily inferred. The result of this special case could be generalized easily to arbitrary static fields. He found that clocks were slowed by gravitational fields and that the now variable speed of light c could stand for the single gravitational potential of static fields.

During this period, Einstein learned a lesson that would be of importance in his later search for the field equations of his general theory of relativity. He found that the conservation laws can circumscribe the range of admissible field equations very powerfully. In 1912 he sought a field equation that would describe how a given source distribution would generate the field.[7] From a natural and simple generalization of Newtonian gravitation theory, he postulated

$$\Delta c = kc\sigma$$

for this equation, where Δ is the Laplacian operator, σ the mass density and k a constant. But he soon found to his dismay that this equation was inconsistent with the equality of action and reaction.[8] In effect, his field equation was inconsistent with the conservation of energy and momentum; it is impossible to construct a gravitational field stress tensor from this field equation and the associated force law. In a protracted discussion of several possible modifications to his theory, Einstein showed how he was forced to a specific and unpalatable resolution: his field equation had to be modified by the subtraction of a particular quantity, $(\mathrm{grad}\, c)^2/2c$, from its left-hand side. He conceded that he resisted this modification to his field equation, for it required him to limit his principle of equivalence to infinitely small regions of space. Since his principle referred only to homogeneous gravitational fields and uniform acceleration, he found the need for such a limitation very puzzling.

With his return to Zürich in August 1912, Einstein took the major step towards his general theory of relativity. There, as he tells us in the foreword to the Czech edition of his popular book on relativity, he had the decisive idea of the analogy with Gauss' theory of surfaces.[9] Also, he began his collaboration with his mathematician friend, Marcel Grossmann, who assisted him with the unfamiliar mathematics required by the new theory. The first product of this collaboration was the *Entwurf* paper, which contained virtually all the essential features of the final general theory of relativity. It was the work of some months only, for Einstein could write to Paul Ehrenfest late in May 1913 that this

7. Einstein, "Lichtgeschwindigkeit" (ref. 6).
8. Einstein, "Zur Theorie" (ref. 6).
9. EA 23 191.

new work was to appear within a few weeks.[10]

The essence of the new theory lay in the fusion of a number of earlier developments. Earlier in 1912, on the basis of the consideration of a rotating coordinate system, Einstein had argued that three dimensional space need no longer remain Euclidean once frames of reference in arbitrary states of motion are introduced.[11] Of course, such frames of reference must be introduced if the principle of relativity is to be extended. Also, it followed from Minkowski's work that one could treat the kinematics of Einstein's new special theory of relativity in terms of the geometry of a pseudo-Euclidean four-dimensional spacetime. In particular, this meant that the familiar Pythagorean formula for the invariant length l in space

$$dl^2 = dx^2 + dy^2 + dz^2$$

is extended to the pseudo-Euclidean formula for the invariant interval s in spacetime,

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2,$$

where x, y, and z are the usual Cartesian spatial coordinates and t the time coordinate. Just as straight lines in Euclidean space are those of minimal l, so the world lines of undeflected particles in spacetime are geodesics, lines of extremal s.

Perhaps the juxtaposition of these two ideas was sufficient to lead Einstein to the central idea of his new theory, the consideration of spacetimes with a more general, non-Euclidean geometry than that of special relativity.[12] Specifically, he considered those in which the interval s is given in terms of the ten components of a symmetric metric tensor, $g_{\mu\nu}$

$$ds^2 = g_{11} dx_1^2 + 2g_{12} dx_1 dx_2 + \cdots + g_{44} dx_4^2 \ .$$

The special attraction in this was that the four spacetime coordinates, x_1, x_2, x_3, and x_4, could be selected arbitrarily, provided the values of the $g_{\mu\nu}$ were adjusted by the appropriate transformation. This associating of spacetime coordinate systems with frames of reference suggests an equivalence of all frames of reference, as demanded by a generalized principle of relativity. Moreover, Einstein and Grossmann could turn to the absolute differential calculus of Christoffel, Ricci, and Levi-Cività, which enabled the writing of the basic laws of the new theory in

10. Einstein to Ehrenfest, 28 May 1913, EA 9 340.

11. Einstein, "Lichtgeschwindigkeit" (ref. 6), 356.

12. See J. Stachel, "Einstein and the rigidly rotating disk," in A. Held, ed., *General relativity and gravitation: A hundred years after the birth of Einstein*, vol. 1 (New York, 1980), 1–15.

a generally covariant form.

Finally, Einstein could interpret the physical significance of the metric tensor by examining certain special and limiting cases. From the requirement that special relativity be a limiting case, it followed that the metric tensor governed the behavior of rods and clocks in space. In addition, Einstein could compare the spacetimes of his new theory with the static gravitational fields of his earlier theory and conclude that non-constancy of the components of the metric tensor corresponded to the presence of a gravitational field. This meant that the metric tensor could be regarded as the generalization of the Newtonian gravitational potential and, in particular, that this single potential was now to be replaced by ten gravitational potentials, the ten components of the metric tensor.

As a part of this reasoning, Einstein made an assumption about the form of static fields that was to cause him a great deal of trouble. In the first section of his part of the *Entwurf* paper, Einstein reviewed the results of his earlier theory of static gravitational fields. Within the second section, he translated some of these results into the formalism of his new theory. He noted that in special relativity—the "usual" theory of relativity, as he put it—the metric tensor degenerates to the simple form

$$\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & c^2 \end{matrix} \quad . \tag{1}$$

where $g_{44} = c^2$ is constant. Recalling his earlier theory, he wrote: "The same type of degeneration appears in static gravitational fields of the type considered just now, only that in these $g_{44} = c^2$ is a function of x_1, x_2, x_3."[13] This formulation compactly expressed several results of his earlier theory that were now being transported intact to his new theory. According to his earlier theory, in static fields the speed of light c is variable and the rate of clocks varies with it. This is encapsulated in the new theory by allowing $g_{44} = c^2$ to be a function of the three spatial coordinates. Einstein had also concluded that three dimensional space would remain Euclidean in these static fields. This now meant that for a proper choice of coordinate system, the remaining components of the metric tensor would retain their constant values as in (1).[14]

13. Einstein and Grossmann (ref. 1), 229.

14. This special case may have been important in the sequence of events in Einstein's transition to the new theory. In an addendum to (ref. 8), he noted that the equations of motion of the theory could be written as a variation principle. Formally the equation he gave is identical to the equation for a geodesic in a spacetime with the metric (1). See J. Stachel, "The genesis of general relativity," in H. Nelkowski, et al., eds., *Lecture notes in physics: Einstein Symposium, Berlin* (Berlin 1979), 428–442, on 433–434.

Einstein did not discover for three years that this last conclusion about static fields was incorrect. It follows from his final theory of November 1915 that static fields are not spatially flat in all but a very few specialized exceptions. But for the time being, Einstein had little reason to doubt this natural extension of the results of his earlier theory. In particular it provided a convenient special case in which the number of gravitational potentials was effectively reduced from ten to a more manageable single potential.

3. THE REJECTION OF THE RICCI TENSOR

The saga of Einstein's search for his general theory of relativity should have ended rapidly and happily here, with the completion of the *Entwurf* paper. And it nearly did. All that he needed was to write the field equations. Einstein opened the critical Section 5 in a familiar and promising way by writing the field equations in the general form

$$\kappa\Theta_{\mu\nu} = \Gamma_{\mu\nu} , \tag{2}$$

where κ is a constant, the source term $\Theta_{\mu\nu}$ is the contravariant stress-energy tensor and, the field term $\Gamma_{\mu\nu}$ is the as yet undetermined gravitation tensor, which is to be built up by differential operations out of the metric tensor, $g_{\mu\nu}$.[15] Since the gravitation tensor is the generalization of the corresponding Newtonian quantity, the Laplacian of the Newtonian gravitational potential $\Delta\phi$, Einstein expected it to be of second order in the derivatives of the metric tensor.

Then Einstein quietly dropped his bombshell. It has proved impossible, he wrote, to find such a differential expression that is a generalization of $\Delta\phi$ and also a generally covariant tensor. Part of his justification is that covariant operations, corresponding to those that generate $\Delta\phi$ out of ϕ in Newtonian theory, yield degenerate results when applied to the metric tensor. The bulk of the justification, however, is a reference to a particular subsection of Grossmann's "Mathematical part" of the paper.

15. I adhere throughout to Einstein's original notation, which is nearly the same as modern notation. However, all indices are written as subscripts, covariant components being indicated by Latin letters, e.g., $g_{\mu\nu}$, and their corresponding contravariant components by Greek letters, e.g., $\gamma_{\mu\nu}$, except for the four contravariant spacetime coordinates, written as x_μ. Gothic letters represent tensor densities. The Einstein summation convention is not used. Greek and sometimes Latin indices vary over 1, 2, 3, and 4, with the 4-component representing the time component. I also follow Einstein's use of the term "tensor" by allowing it to describe quantities covariant under limited as well as under arbitrary coordinate transformations. Between 1912 and 1915 Einstein modified his notation gradually until it achieved the form now standard.

Grossmann noted that the prominent position of the "Christoffel four index symbol," that is, the fourth-rank Riemann curvature tensor, would lead us to expect that its second rank contraction, now called the "Ricci tensor," would be a natural candidate for the gravitation tensor. "However [he wrote] it turns out that this tensor does *not* reduce to the expression $\Delta\phi$ in the special case of an infinitely weak, static gravitational field."[16] With this, both Grossmann and Einstein dropped the question of constructing generally covariant field equations out of the Riemann tensor and, apparently, out of any expression of second order in the derivatives of the metric tensor.

This was a catastrophe. A continued focus on the Riemann curvature tensor would have set them on a royal road to the generally covariant field equations of the final theory. The selection of the Ricci tensor as the gravitation tensor would have given them these equations in the source-free case. The discovery of the additional term necessary for the general case would have been but a small step, as it proved to be for Einstein in November 1915.[17]

It is clear from Grossmann's brief comment that the decision to turn away from the Ricci tensor resulted from a problem preventing recovery of a Newtonian limit from the field equations concerned. But he gives virtually no clues to the way they attempted this recovery or to the exact nature of the problem they encountered. Fortunately, in Section 5 of their paper, Einstein went to some pains to establish the form that an appropriate generalization of $\Delta\phi$ must have and, correspondingly, the form that the gravitation tensor must take if it is to reduce to this expression in appropriate cases. The required form $\Gamma_{\mu\nu}$ is given as

$$\Gamma_{\mu\nu} = \sum_{\alpha\beta} \frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) + \left\{\quad\right\}, \tag{3}$$

where the curly brackets indicate terms that drop out in first order when the derivatives of $g_{\mu\nu}$ are small. In brief, this is justified by noting that only the highest order terms in equation (3) remain in the weak-field case and reduce to an expression of the form

$$\Gamma_{\mu\nu} = -\left(\frac{\partial^2\gamma_{\mu\nu}}{\partial x_1^2} + \frac{\partial^2\gamma_{\mu\nu}}{\partial x_2^2} + \frac{\partial^2\gamma_{\mu\nu}}{\partial x_3^2} - \frac{1}{c^2}\frac{\partial^2\gamma_{\mu\nu}}{\partial x_4^2}\right)$$

16. Einstein and Grossmann (ref. 1), 256–257.

17. In modern notation, these equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = kT_{\mu\nu}$$

where $R_{\mu\nu}$ is the Ricci tensor with contraction R, and $T_{\mu\nu}$ is the stress-energy tensor. The left-hand side of the equation is now called the "Einstein tensor".

provided that the components of the weak field metric differ from those of the special relativistic metric of form (1) by infinitely small quantities. (The right-hand side of the preceding equation will be written as $\Box\gamma_{\mu\nu}$.) From this, $\Gamma_{\mu\nu}$ reduces to a satisfactory Newtonian form, with

$$\Gamma_{44} = -\Delta\gamma_{44}$$

as the only nonvanishing term, in the case of a static field, in which g_{44} alone is variable.

This last case corresponds exactly to the special case considered by Grossmann, that of "an infinitely weak, static gravitational field." This promises to give us an immediate explanation of Einstein and Grossmann's rejection of the Ricci tensor as a gravitation tensor. For a direct inspection of the form of the Ricci tensor, written out explicitly in terms of derivatives of the metric tensor, shows that it fails to satisfy Einstein's condition (3) above. In fact the Ricci tensor contains four second derivative terms in the metric tensor which are of first order of smallness when the derivatives of the metric tensor are small. Only one of these corresponds to the one in condition (3).

However, the elimination of these three other second derivative terms can be achieved readily in the process of recovery of the Newtonian limit. As I pointed out earlier, generally covariant field equations, such as those based on the Ricci tensor, hold in all coordinate systems, whereas the equations of Newtonian theory do not. So in the process of recovering Newtonian theory we must restrict the set of coordinate systems under consideration, by requiring, for example, the satisfaction of additional constraints. In particular, we could consider coordinate systems in which the "harmonic" coordinate conditions

$$g^{\mu\nu}\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix} = 0 \tag{4}$$

are satisfied (the curly bracket is the Christoffel symbol of the second kind). In such coordinate systems, the three additional second derivative terms vanish and the Ricci tensor reduces to the form required by condition (3). The recovery of the expected Newtonian limit then follows readily from the consideration of weak fields in these coordinate systems.

Thus if we wish to explain Einstein and Grossmann's turning away from the Ricci tensor as a result of their inability to reduce it to the form required by condition (3), we must assume that neither had sufficient facility or familiarity with tensor calculus to be able to find such a condition as (4). Or worse, we might assume that neither was even aware of his freedom to apply coordinate conditions.

It is difficult to believe that both Einstein and Grossmann could have been ignorant of this freedom and that their ignorance should have persisted over several years of intense study and reflection. Nevertheless, several commentators believe it.[18] Their approach is made at least provisionally viable by the fact that neither Einstein nor Grossmann made any explicit acknowledgement of their freedom to use such coordinate conditions in their publications at the time of the *Entwurf* paper. The approach has significant attraction: it enables a very simple explanation of the apparent fallacy in Einstein's later "hole" argument against the physical admissibility of all generally covariant field equations. Einstein was unaware, it is supposed, that the imposition of coordinate conditions does not alter the physical content of the laws of his theory; rather it affects only their mathematical form in restricting them to certain coordinate systems. We shall see later that Einstein's "hole" argument can be approached by supposing that Einstein was unaware of a closely related result. That is, that a change in the coordinate system will affect only the mathematical functions representing the physical quantities of his theory, without actually changing the physical quantities themselves. That is, they will "look different" to us, but, of course, remain physically unchanged by our change of viewpoint.[19]

That such an account of Einstein's three-year struggle has become widespread is not surprising. Three out of four of Einstein's presentations of the "hole" argument are extremely brief, and admit the standard interpretation. Moreover, the evidence necessary to acquit Einstein of the charge of ignorance of his freedom to apply coordinate conditions is neither as accessible nor, at present, as readily available as his publications from this period. Nevertheless, it seems that the simplifications of the standard account could have been avoided. I do not mean that we should assume that Einstein never made mistakes. In the course of this paper we shall see him make and correct quite a few mistakes. But we could have been more wary of accounts that try

18. See C. Lanczos, "Einstein's path from special to general relativity," in L. O'Raifeartaigh, ed., *General relativity: Papers in honour of J. L. Synge* (Oxford 1972), 5–19, on 13–14; A. Pais, *Subtle is the Lord: The science and life of Albert Einstein* (Oxford 1982), 221–223, 243–244; V. P. Vizgin and Ya. A. Smorodinskii, "From the equivalence principle to the equations of gravitation," *Soviet Physics Uspekhi, 22* (1979), 489–513, on 501–502; Earman and Glymour (ref. 1), 256–257. Mehra discusses the rejection of the Ricci tensor in terms of Einstein's difficulties in finding a generally covariant conservation law, but these difficulties became important only at a slightly later stage. J. Mehra, *Einstein, Hilbert and the theory of gravitation* (Dordrecht, 1974), 11–12.

19. Besides the items in ref. 18, see B. Hoffmann, "Einstein and tensors," *Tensor, 26* (1972), 157–162, on 161, and "Some Einstein anomalies," in G. Holton and Y. Elkana, eds., *Albert Einstein: Historical and cultural perspectives* (Princeton, 1982), 91–105, on 100–102; E. Zahar, "Einstein, Meyerson and the role of mathematics in physical discovery," *British journal for the philosophy of science, 31* (1980), 1–43, on 31–33.

to explain the errors of three years of intense searching as beginners' blunders.

The first sustained attempt to write an account of these episodes which would not convict Einstein and Grossmann of such simple and fundamental errors was made by Stachel.[20] He pointed out that the special case that Grossmann considered was not just that of an infinitely weak field, but that of an infinitely weak, *static* field. He also noted that Einstein's expectations for the form of static fields were inconsistent with his final theory, in the way we have seen here. If Grossmann or Einstein calculated the components of the Ricci tensor $G_{\mu\nu}$ in infinitely weak fields of this special type, it is easy to reconstruct what they found. If the metric of the field is of the form (1) and the derivatives of c are small, it follows that the Ricci tensor's only nonvanishing components of the first order are

$$G_{44} = \frac{1}{2}\Delta c^2 = \frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)c^2 \qquad (5)$$

$$G_{ij} = -\frac{1}{2c^2}\frac{\partial^2 c^2}{\partial x_i \partial x_j}$$

where i and j vary over 1, 2, and 3 only. The G_{44} term looks promising, but the remaining terms are disastrous. In the source free case, the field equations take the form

$$G_{\mu\nu} = 0 \ .$$

With equations (5), this yields the unacceptable conclusion that c^2 can vary at most linearly with the spatial coordinates. Thus, if we assume that static gravitational fields have the form (1), as Einstein believed, then it follows that the Ricci tensor is unacceptable as the gravitation tensor.

Through this conjecture, we can now see that Einstein and Grossmann's rejection of the Ricci tensor need not be explained in terms of a simple error, but that it may have resulted from a deep-seated misconception about the nature of static fields. Some problems remain, however. Beyond the brief remark of Grossmann's cited earlier, there is no direct evidence that Einstein or Grossmann ever actually performed the calculation described. Of greater importance, it still does not tell us whether Einstein and Grossmann were aware of their freedom to apply coordinate conditions. Of course the assumption that the static field in question has a metric with components of the form (1) contains an *implicit* coordinate condition. For the coordinate system

20. Stachel (ref. 3).

must be chosen in such a way that the components of the metric do in fact adopt the required form. But were they *explicitly* aware of this in 1912 and 1913?

4. THE ZÜRICH NOTEBOOK

In the Einstein Archive is a notebook originally cataloged as containing notes for Einstein's lectures at the University of Zürich in the period 1909–11. However, the contents of the notebook, all written in Einstein's hand, are not lecture notes but scratch-pad calculations.[21] The subject matter includes statistical physics, thermodynamics, and basic principles of the four-dimensional representation of electrodynamics. The major part of the notebook, which extends from pages 5 to 29, belongs to 1912–13, for it contains calculations made by Einstein during his work on the *Entwurf* paper. These calculations are accompanied by virtually no explanatory text. Fortunately their import can generally be deciphered.

The section in question begins with the heading *Gravitation* and contains various formulae, including generally covariant expression of the equations of motion of a point mass and the laws of conservation of energy and momentum. Einstein's treatment here closely corresponds to that of the early sections of the *Entwurf* paper. He investigates various basic questions in his new theory, including the properties of general and rotating coordinate transformations. This part also contains a study of generalized d'Alembertian operators and the formation of associated expressions out of the metric tensor and its derivatives. These appear to be some of Einstein's earliest speculations on the problem of constructing field equations for his new theory. He writes next to one term "vermutlicher Gravitationstensor"—"supposed gravitation tensor."

After these preliminaries, which stop at page 14, Einstein began what seem to be his earliest attempts to construct a generally covariant gravitation tensor out of the Riemann curvature tensor. He wrote out this fourth rank curvature tensor explicitly, with the notation "Grossmann/Tensor vierter/Mannigfaltigkeit"—"Grossmann/tensor of fourth/rank." This, of course, suggests that this expression was provided for him by Grossmann, as we would expect at this early stage of their collaboration. So it is not surprising that it matches Grossmann's formula (43) in the *Entwurf* paper exactly—even including the choice of letters used to label the indices. The four second derivative terms of the expression Einstein wrote down were:

21. AE 3 006. John Stachel had already noted its mislabelling.

$$\frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x_k \partial x_l}+\frac{\partial^2 g_{kl}}{\partial x_i \partial x_m}-\frac{\partial^2 g_{il}}{\partial x_k \partial x_m}-\frac{\partial^2 g_{km}}{\partial x_i \partial x_l}\right). \tag{6}$$

Einstein then contracted this once over k and l to form the Ricci tensor and proceeded to calculate its first derivative terms explicitly. Then the page was ruled off and Einstein wrote:

$$"\sum_k \left(\frac{\partial^2 g_{kk}}{\partial x_i \partial x_m}-\frac{\partial^2 g_{ik}}{\partial x_k \partial x_m}-\frac{\partial^2 g_{mk}}{\partial x_k \partial x_i}\right) \text{ ought to vanish."} \tag{7}$$

By comparison with equation (6), we can see that these three terms are taken from the explicit expression for the second derivative terms of the Ricci tensor.[22]

The point of Einstein's remark seems quite clear. The expression in (7) "ought to vanish" exactly in case Einstein were to require the Ricci tensor to have the form (3). Since this expression does not vanish in general, Einstein had found that the Ricci tensor does not have the form (3), which he required of a gravitation tensor. So far, what we have seen here is entirely in accord with Einstein's presumed ignorance of his freedom to apply coordinate conditions, for he had not mentioned the coordinate conditions that could be used to make these three terms vanish.

Einstein continued with an extended calculation of the explicit form of the Riemann curvature scalar. He had some difficulty working with all the terms since he dropped a factor of 1/4. But, under the constraint that the determinant of the metric tensor, written as G, should be equal to unity, he came to a final result on page 16. There his purpose becomes clear, for he tried to divide the scalar into the contraction of two tensors, $\sum_{ik} g_{ik} \mathfrak{J}_{ik}$. This $\mathfrak{J}_{ik}$ would presumably again be a candidate for the gravitation tensor. He then turned to a calculation of the contravariant components of the Ricci tensor. This calculation is broken off early as "zu umständlich," "too involved."

So far, Einstein seems to have made no real progress since he discovered the failure of the Ricci tensor to take on the form (3). He introduced a new result of crucial importance, however, on page 19. This is the condition:

$$\gamma_{kl}\begin{bmatrix} kl \\ i \end{bmatrix} = 0\,. \tag{8}$$

Five pages earlier Einstein defined the term in square brackets to be the

22. That Einstein sums over k in such terms as g_{kk} indicates that he is considering a weak-field case in a coordinate system with an imaginary x_4 ("time") coordinate and in which the metric is Minkowskian and represented by the unit matrix to zeroth order.

Christoffel symbol of the first kind. So, in assuming summation as implied over repeated indices, we recognize equation (8) as the harmonic coordinate condition (4). Einstein proceeded to confirm that this condition reduces the Ricci tensor to the required form of equation (3). On pages 19–22, Einstein studied the behaviour of the weak field that would follow from taking the Ricci tensor as a gravitation tensor in harmonic coordinates. He focused on the energy-momentum conservation law and the construction of a gravitational field stress-energy tensor.

Whatever he found there must not have pleased him greatly, for on page 22 he took a completely new approach to the problem of constructing a generally covariant gravitation tensor from the Riemann curvature tensor. Under the heading "Grossmann" he wrote an expression for the covariant Ricci tensor, denoted by $\mathfrak{T}_{il}$, which exactly matches the contraction of Grossmann's expression (44) for the Riemann curvature tensor in the *Entwurf* paper. Einstein declared:

If G is a scalar, then $\dfrac{\partial \, lg\sqrt{G}}{\partial x_i} = \mathfrak{T}_i$ tensor 1st rank;

$$\mathfrak{T}_{il} = \underbrace{\left(\frac{\partial \mathfrak{T}_i}{\partial x_l} - \sum \begin{Bmatrix} il \\ \lambda \end{Bmatrix} \mathfrak{T}_\lambda\right)}_{\text{tensor 2nd rank}} - \underbrace{\sum_{k\lambda}\left(\frac{\partial \begin{Bmatrix} il \\ \lambda \end{Bmatrix}}{\partial x_k} - \begin{Bmatrix} ik \\ k \end{Bmatrix}\begin{Bmatrix} l\lambda \\ k \end{Bmatrix}\right)}_{\text{supposed gravitation tensor.} \ \mathfrak{T}^{x}_{il}} \tag{9}$$

The interpretation of this equation is made very easy by the fact that Einstein's procedure corresponds exactly to his first method of constructing a generally covariant gravitation tensor when he returned to the problem in late 1915.[23] There he showed that if one restricts oneself to coordinate transformations with a determinant of one, then G becomes a scalar, the first term in the expansion of the Ricci tensor is itself a tensor as marked and thus the second term, which is taken as the gravitation tensor, is also a tensor.

This close correspondence to the content of Einstein's later paper is in itself a fascinating discovery. It also provides an unexpected confirmation of Einstein's claim in the introduction to that paper that, three years earlier, he and Grossmann "had actually already come quite close to the solution of the problem given in the following."[24] The

23. Einstein, "Zur allgemeinen Relativitätstheorie," AW, *Sb*, 1915, 778–786. Compare this equation (9) with Einstein's equations (42), (43), and (44) below.

24. Einstein (ref. 23), 778; cf. Einstein to Sommerfeld, 23 Nov 1915 in A. Hermann, ed., *Albert Einstein—Arnold Sommerfeld Briefwechsel* (Basel/Stuttgart, 1968), 23–26.

reason that Einstein elected to split up the Ricci tensor in this way appears on page 22. There he showed the application of the coordinate condition

$$\sum_k \frac{\partial \gamma_{k\alpha}}{\partial x_k} = 0 \tag{10}$$

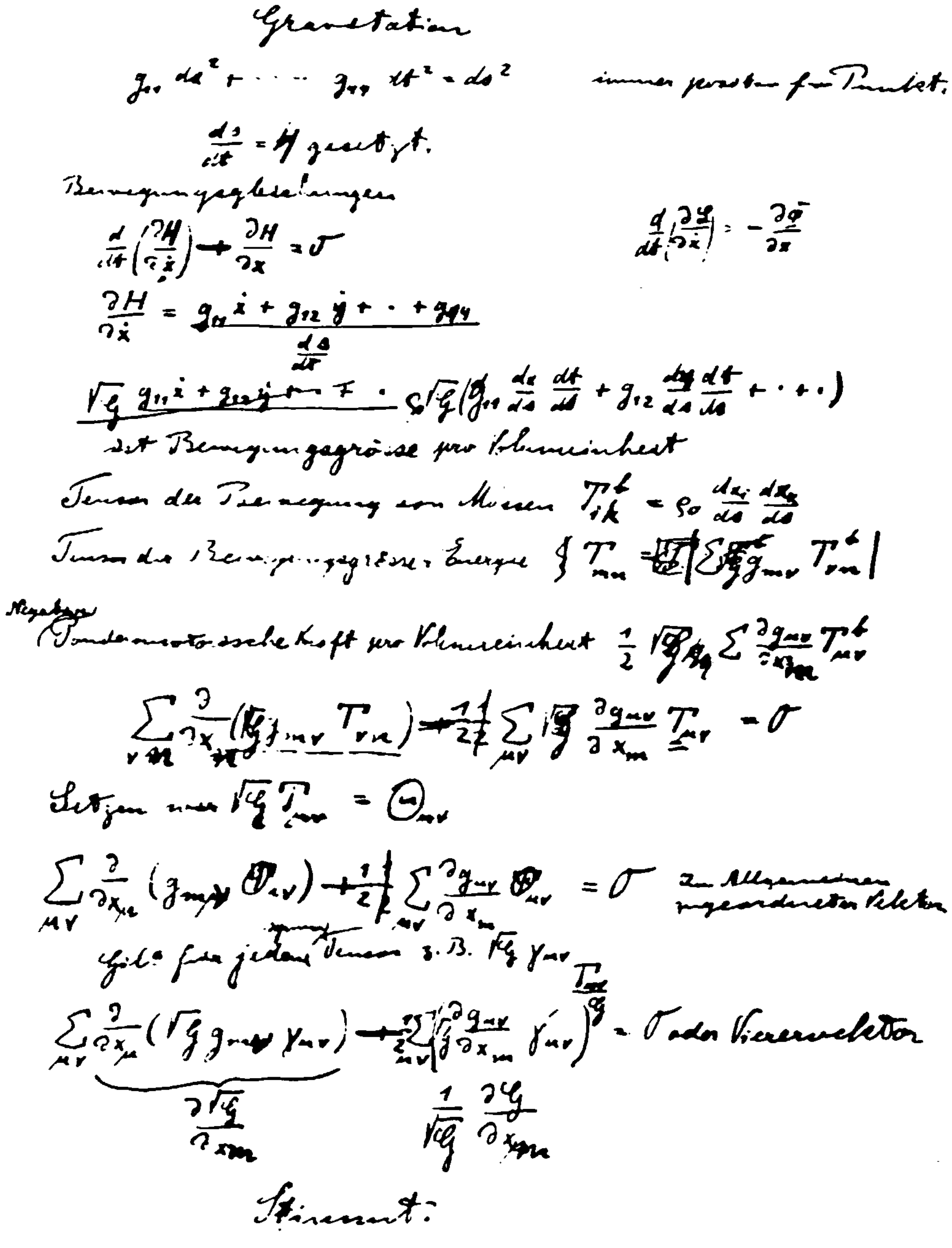
leads to the reduction of the new gravitation tensor to the form required in equation (3).

Einstein next investigated the behavior of his new gravitation tensor and coordinate condition particularly in connection with the energy-momentum conservation law. He constructed what I take to be the (coordinate) divergence of the gravitational field stress-energy tensor, which, he confirmed, vanishes in Minkowski spacetime when viewed from rotating coordinates. Then, quite abruptly, he broke off the search for generally covariant gravitation tensors. On page 26, under the heading "Ableitung der Gravitationsgleichungen"—"derivation of the gravitation equations"—Einstein wrote out, or perhaps transcribed, a tight summary of the derivation of identity (12) of his part of the *Entwurf* paper. This comprises a major part of the derivation of the *Entwurf* field equations. Its appearance here, and the want of these equations earlier in the sequence of the bound pages of the notebook, indicate that the *Entwurf* field equations and their derivation came after the attempts to construct an acceptable generally covariant gravitation tensor outlined above.[25]

The problem of coordinate conditions

From Einstein's notebook we learn that his search for a generally covariant gravitation tensor was dominated by the requirement that it take the form given in equation (3). We also see that he was already quite aware of his freedom to apply coordinate conditions to achieve this form and that he discovered *two* suitable gravitation tensors that reduced to the required form with appropriate coordinate conditions.

25. I briefly review the internal evidence for dating the material in the notebook to the period of writing the *Entwurf* paper. Its notation, especially the use of $\gamma_{\mu\nu}$ for the contravariant components of the metric tensor, date it before mid-1914. Its generally elementary content and the similarity of its treatment to that of the *Entwurf* paper place it at the very beginning of Einstein's work on the theory, as does the repeated crediting of the formula for the Riemann curvature tensor to Grossmann. Note also the omission of such concepts as "adapted coordinates" and of the use of variational techniques that became characteristic of Einstein's work on the theory after 1913.



Gravitation

$g_{11} dx^2 + \cdots g_{44} dt^2 = ds^2$ immer positiv für Punkt.

$\frac{ds}{dt} = H$ gesetzt.

Bewegungsgleichungen

$$\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) - \frac{\partial H}{\partial x} = 0 \qquad \frac{d}{dt}\left(\frac{\partial \varphi}{\partial \dot{x}}\right) = -\frac{\partial \varphi}{\partial x}$$

$$\frac{\partial H}{\partial \dot{x}} = \frac{g_{11}\dot{x} + g_{12}\dot{y} + \cdots + g_{14}}{\frac{ds}{dt}}$$

$$\sqrt{g}\,\frac{g_{11}\dot{x} + g_{12}\dot{y} + \cdots}{} = \rho_0\sqrt{g}\left(g_{11}\frac{dx}{ds}\frac{dt}{ds} + g_{12}\frac{dy}{ds}\frac{dt}{ds} + \cdots\right)$$

ist Bewegungsgrösse pro Volumeinheit

Tensor der Bewegung von Massen $T_{ik} = \rho_0 \frac{dx_i}{ds}\frac{dx_k}{ds}$

Tensor der Bewegungsgrösse & Energie $\mathfrak{T}_{mn} = \sqrt{g}\sum g_{m\nu} T_{\nu n}$

Negative Ponderomotorische Kraft pro Volumeinheit $\frac{1}{2}\sqrt{g}\sum \frac{\partial g_{\mu\nu}}{\partial x_m} T_{\mu\nu}$

$$\sum_{\nu n}\frac{\partial}{\partial x_n}\left(\sqrt{g}\, g_{m\nu} T_{\nu n}\right) - \frac{1}{2}\sum_{\mu\nu}\sqrt{g}\,\frac{\partial g_{\mu\nu}}{\partial x_m} T_{\mu\nu} = 0$$

Setzen wir $\sqrt{g}\,T_{\mu\nu} = \Theta_{\mu\nu}$

$$\sum_{\mu\nu}\frac{\partial}{\partial x_\mu}\left(g_{m\nu}\Theta_{\mu\nu}\right) - \frac{1}{2}\sum_{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x_m}\Theta_{\mu\nu} = 0$$ zu allgemeinen ungeordneten Vektor

Gilt für jeden Tensor z.B. $\sqrt{g}\,\gamma_{\mu\nu}$

$$\sum_{\mu\nu}\frac{\partial}{\partial x_\mu}\left(\sqrt{g}\, g_{m\nu}\gamma_{\mu\nu}\right) - \frac{1}{2}\sum_{\mu\nu}\sqrt{g}\,\frac{\partial g_{\mu\nu}}{\partial x_m}\gamma_{\mu\nu} = 0$$ oder Vierervektor

$$\frac{\partial\sqrt{g}}{\partial x_m} \qquad \frac{1}{\sqrt{g}}\frac{\partial g}{\partial x_m}$$

Stimmt!

FIG. 1 Zürich notebook, page 5 (right-hand side). The subject of gravitation is introduced with some elementary results from the *Entwurf* theory. Compare ref. 1, 229, 232. These results include the equations of motion of a unit point mass, written in a Hamiltonian form, and the identification of various "quantities of motion," which are combined to yield the law of conservation of energy and momentum, written as the vanishing of the covariant divergence of the stress energy tensor.

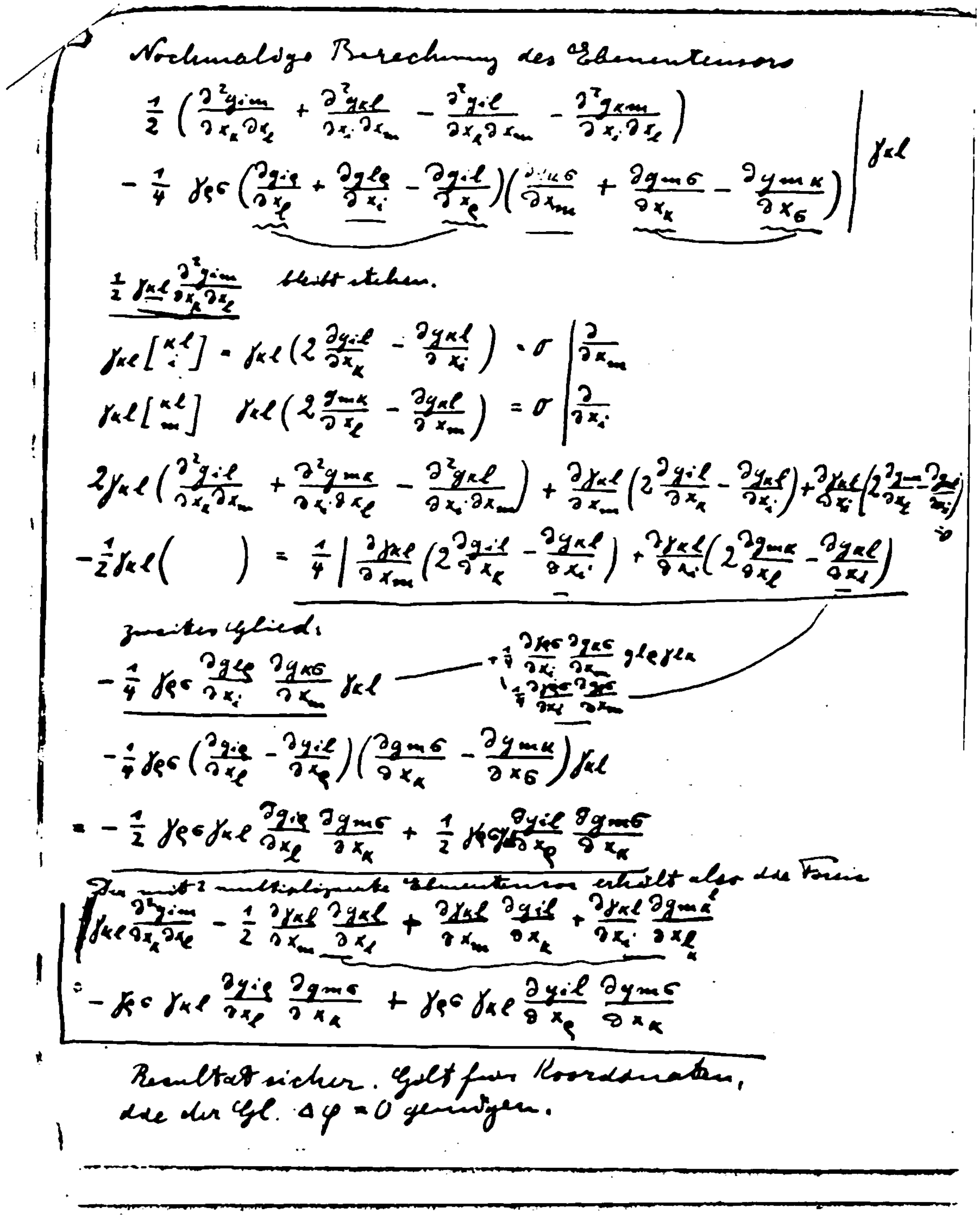


FIG. 2 Zürich notebook, page 19 (left-hand side). In the first three lines, Einstein shows that he intends to contract the Riemann curvature tensor with $\gamma_{\kappa l}$ leaving only a d'Alembertian-like term in second derivatives of the metric. He succeeds easily with the harmonic condition $\gamma_{\kappa l}\begin{bmatrix}\kappa l \\ i\end{bmatrix} = 0$. Of the final result, given in the last two lines, he remarks, "Result certain. Holds for coordinates, which satisfy the eq[uation] $\Delta\phi = 0$."

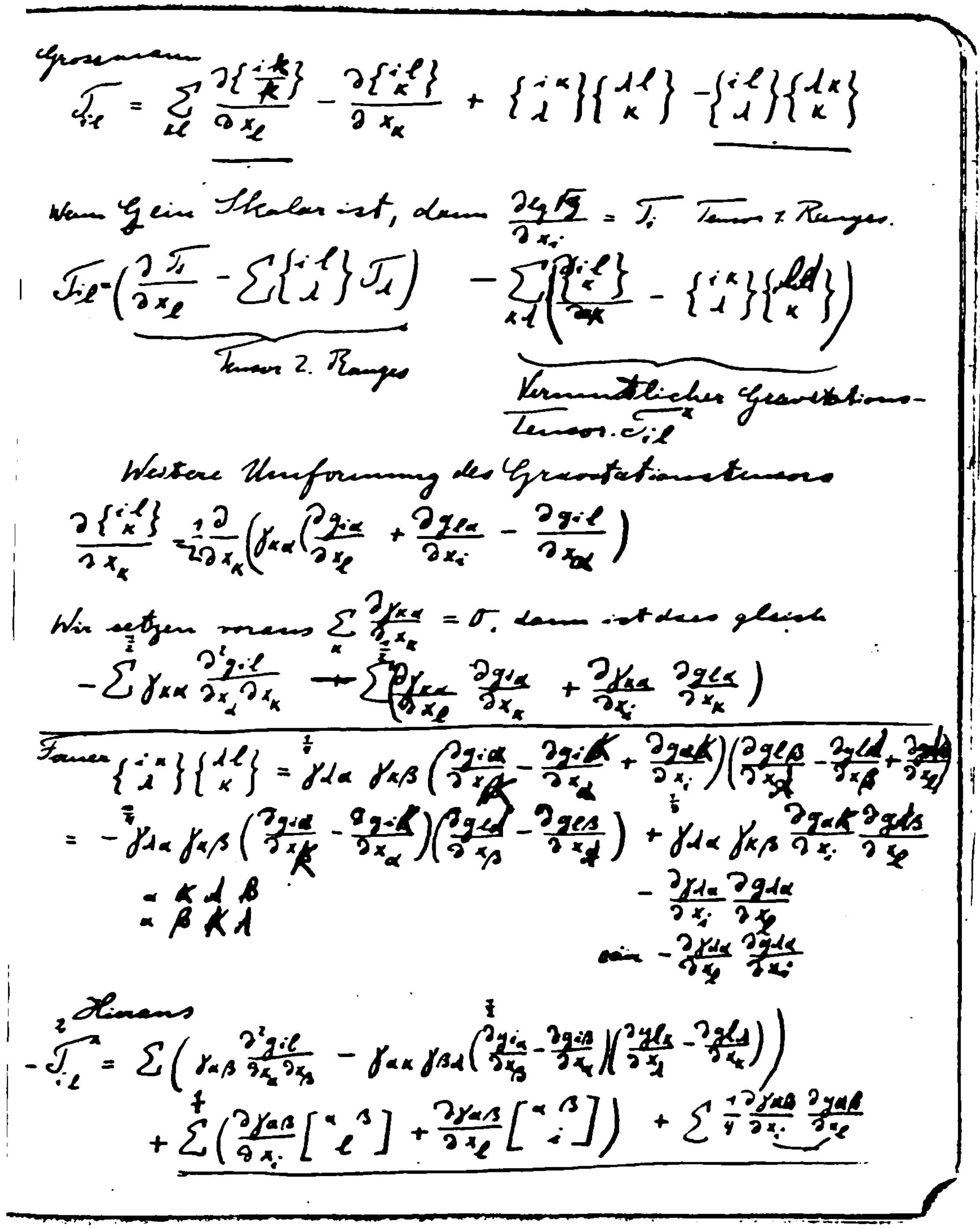


FIG. 3 Zürich notebook, page 22 (right-hand side). In the second equation, Einstein breaks up the Ricci tensor into two parts, as described in the text, the second being his presumed gravitation tensor. Application of the coordinate condition $\sum_{\kappa} \partial\gamma_{\kappa\alpha}/\partial\kappa_{\alpha} = 0$ enables reduction of this tensor to a form whose only second derivative term in the metric is of the required d'Alembertian form. The complete reduced form of the gravitation tensor is at the bottom of the page.

In solving one problem, however, we have created another. Given that Einstein had these results prior to the completion of the *Entwurf* paper, why did he reject them and construct field equations of severely limited covariance? Unfortunately, the question cannot be answered decisively on the basis of the material at hand. However, we can note that there are two persistent characteristics of the calculations in which Einstein appears to test out his proposed gravitation tensors. First, they are concerned with the conservation of energy-momentum and the construction of a gravitational field stress-energy tensor. Second, they relate to a few special cases: weak and static fields and spacetime viewed from rotating coordinates. These are sufficient clues to enable a reasonable conjecture why Einstein rejected these gravitation tensors. I conjecture that his reasons reduce to two basic points:

- Both of the coordinate conditions that Einstein considered fail in at least one of the special cases in which he would have expected them to hold.
- Einstein believed that the derivation of the *Entwurf* field equations gave a unique result, which disagreed with both of the gravitation tensors proposed in the notebook, even after the application of the coordinate conditions.

I will return to the second point at the end of this section.

To begin analysis of the first point, note that Einstein's most favored special case in this period is that of the static gravitational field, which he assumed to have the form given in (1), where c^2 is a function of the spatial coordinates. Einstein certainly considered this special case in the notebook (on pages 6 and 21) and after his examination of the Ricci tensor in harmonic coordinates. Now such a field in the form (1) does not satisfy the harmonic coordinate condition. So the Ricci tensor would appear to Einstein not to reduce to the required form of (3) in this most basic of cases. Einstein would have regarded this failure as a major defect—perhaps in itself sufficient basis for Grossmann's claim that the Ricci tensor does not reduce to $\Delta\phi$ in the case of a weak, static field. It is not surprising that Einstein should then have continued to search for another generally covariant gravitation tensor and proceed to arrive (on page 22) at $\mathfrak{T}^{x}_{il}$. The coordinate condition associated with this tensor, equation (10), is satisfied in Einstein's static field, with the metric (1).

In the last relevant part of the notebook (page 24 ff.), Einstein reintroduced the case of a Minkowski spacetime viewed from Cartesian coordinates in uniform rotation. We know that Einstein regarded this case as crucial to the general relativity of motion, so he expected his field equations to hold in such a case.[26] We shall see that Einstein dealt

26. Einstein, "Die formale Grundlage der allgemeinen Relativitätstheorie," AW, *Sb*, 1914, 1030–1085, on 1067–1068 (received 29 Oct 1914).

with this case in a manner that suggests that he expected the gravitation tensor to retain the form (3). This assumption was natural. For he readily, but mistakenly, came to believe that his *Entwurf* field equations held in such a rotating system—and that they have a gravitation tensor of the form (3).[27] Now coordinate condition (10) does not hold in this case, nor does the harmonic condition, another argument for Einstein against the Ricci tensor. If Einstein approached the problem in the way set out here, he would inevitably be drawn to reject $\mathfrak{T}^{x}_{il}$ as a gravitation tensor.

This interpretation is consistent with a letter Einstein wrote to Sommerfeld late in 1915, in which he told him of his success in discovering generally covariant field equations:[28]

> One can eminently simplify the whole theory by choosing the reference system so that $\sqrt{-g} = 1$. Then the equations take on the form
>
> $$-\sum_{l} \frac{\partial \left\{ {im \atop l} \right\}}{\partial x_l} + \sum_{\alpha\beta} \left\{ {i\alpha \atop \beta} \right\} \left\{ {m\beta \atop \alpha} \right\} = -\kappa \left(T_{im} - \frac{1}{2} g_{im} T \right).$$
>
> I had already considered these equations three years ago with Grossmann (up to the second term on the right hand side). But then I had come to the result that they did not yield Newton's approximation, which was erroneous.

This confirms the dating of the material in the Zürich notebook, for the field equation described here is exactly the one that would follow from placing $\mathfrak{T}^{x}_{il}$ in the general field equation (2). The rejection of this equation, as Einstein tells us here, was due to a mistaken belief that it did not provide Newtonian theory as an approximation. We know that he was aware then that the coordinate condition (10) led to a reduction of $\mathfrak{T}^{x}_{il}$ to the expected form in the weak-field case. We can only conclude that he had some objection to the coordinate condition itself.

In a letter to Paul Hertz, probably written in 1915, Einstein mentioned that he had had serious difficulties with coordinate conditions.[29] The general context is the problem of "adapted" coordinates, which he introduced with Grossmann in 1914. He singled out coordinate condition (10) as one coordinate that he had considered, but, unfortunately, he did not give further details on its use or the reasons for its rejection.

27. A. Einstein to M. Besso, *ca.* Mar 1914, in P. Speziali, ed., *Albert Einstein—Michele Besso: Correspondence, 1903–1955* (Paris, 1972), 52–53.

28. A. Einstein to A. Sommerfeld, 23 Nov 1915 (ref. 24). T is the trace of T_{im}.

29. A. Einstein to P. Hertz, 22 Aug 1915?, EA 12 202.

The reduction of the gravitational potentials

The last of the errors responsible for Einstein's rejection of the Ricci tensor as the gravitation tensor was an insistence that the gravitation tensor must adopt the form (3) in rotating coordinate systems after the application of the appropriate coordinate condition. This was the least supportable of Einstein's misconceptions. Thus it is not surprising that its discovery came at an early stage in Einstein's 1915 return to general covariance.

The other errors were products of an extremely coherent but nonetheless mistaken view of static and weak fields. We have seen how Einstein came to believe that the metric degenerated in static fields to the form of (1), where $g_{44} = c^2$ is a function of the spatial coordinates and, due to the constancy of the remaining components of the metric, three dimensional space is flat. This result seemed to be anchored firmly in the principle of equivalence. In his earlier theory, Einstein had invoked the principle to produce a homogeneous field by transforming to a coordinate system in uniform rectilinear acceleration. In terms of the *Entwurf* theory, the field that resulted had the degenerate form given above, with g_{44} now a *linear* function of the spatial coordinates. It seemed entirely unremarkable to generalize from this very special case to the case of more general static fields through the relaxation of this condition, which now allowed g_{44} to be an arbitrary function of the spatial coordinates. But this last generalization turned out to be inadmissible from the standpoint of the final general theory of relativity and led Einstein to a view of static fields inconsistent with that theory. For according to it, static fields are not spatially flat in all but very few exceptions. At the time, however, the view that arbitrary static fields are spatially flat was simple and attractive, for it promised to reduce the number of gravitational potentials from the unwieldy ten of the general case to a more manageable and familiar single potential, g_{44}.

Einstein might well have been able to recover from this mistake quite rapidly were it not for an unfortunate coincidence. Einstein required that the gravitation tensor reduce to the form (3) in appropriate coordinate systems. This amounts to requiring that the gravitation tensor reduce to the d'Alembertian of the metric tensor, $\Box g_{\mu\nu}$, in the weak-field case. The naturalness of this requirement is virtually unchallengeable. It guarantees at least Lorentz covariance for the theory in the weak-field case; it guarantees that this weak-field theory will generalize Newtonian theory in much the same way as electrodynamics generalizes electrostatics; and it does both in about the simplest way possible.

However, when combined with the assumption that the field equations have the form (2), this natural requirement led to results inconsistent with Einstein's final general theory of relativity. For it led to the weak-field equations

$$\Box g_{\mu\nu} = \kappa T_{\mu\nu} , \quad (11)$$

whereas his final theory yielded

$$\Box g_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) \quad (11')$$

in similar circumstances.[30] Unfortunately for Einstein, it turned out that his weak field equation (11) could be solved in appropriate cases to yield exactly the static fields he expected from his earlier considerations. Equations (11′), however, do not yield such solutions.

As Einstein showed a little later in 1913, in the case of a static weak field, whose only source to first order is a pressureless, static dust cloud of density ρ_0, weak-field equations (11) reduce to

$$\Delta g^{*}_{44} = \kappa c^2 \rho_0$$

$$\Delta g^{*}_{\mu\nu} = 0 \quad (\text{all other } \mu,\nu)$$

where the asterisks denote deviations from Minkowskian values.[31] Provided the second equation holds everywhere in spacetime, it can be solved to yield constant values for all the $g_{\mu\nu}$, excepting g_{44}. The first equation then solves to yield a g_{44} that behaves exactly like the familiar Newtonian potential. If the background Minkowskian metric is taken to have the values of equation (1), then this solution amounts to the recovery of exactly the metric that Einstein expected in static fields.

Such a simple reduction to a single gravitational potential does not occur in the final theory. However, to Einstein at this early stage, it would have seemed quite natural, for some such reduction had to occur in the process of recovering Newtonian gravitation theory. And, of course, this simple reduction agreed exactly with the type of reduction

30. Compare with Einstein's own later writing of the weak-field equations of his final theory in A. Einstein, *The meaning of relativity* (London, 1976), 83. After reduction with the harmonic coordinate condition, they are

$$\frac{\partial^2 \gamma_{\mu\nu}}{\partial x_\alpha^2} = 2\kappa(T_{\mu\nu} - g_{\mu\nu} T) ,$$

where $\gamma_{\mu\nu}$ are the weak-field deviations of the components of the metric tensor from their Minkowskian values. Here Einstein began with the presumption that all the $\gamma_{\mu\nu}$ were of the same order. He later rejected this presumption. This altered the further development of these weak-field equations and their relationship to the Newtonian case. See A. Einstein, L. Infeld, and B. Hoffmann, "The gravitational equations and the problem of motion," *Annals of mathematics, 39* (1938), 65–100.

31. Einstein, "Zum gegenwärtigen Stande des Gravitationsproblems," *Physikalische Zeitschrift, 14* (1913), 1249–1266 (lecture of 23 Sep 1913), 1259.

Einstein believed would happen in static fields on the basis of a quite separate consideration, the principle of equivalence.

This reduction of the number of gravitational potentials in the weak-field case followed also from the form of equation (11) in weak fields that need not be static through an order-of-magnitude argument. If the source of the field is a dust cloud, which now need not be static, then the (4,4) term of the stress-energy tensor on the right-hand side of equation (11) is, typically, significantly larger than all its other components. Transferring this property to the field term on the left-hand side, it follows that the only first order deviations from constant values in the components of the metric tensor are in the g_{44} term.[32]

This argument cannot be used on the weak field equations (11′) of Einstein's final theory. In this final theory, more components of the metric than g_{44} are variable. However, only the g_{44} component appears in the equations of motion of a slow moving point mass. From the point of view of the final general theory of relativity, this alone enables Newtonian theory to account successfully for the motion of a slow moving point mass in terms of one gravitational potential only. Einstein commented to Besso late in 1915 on this remarkable feature of the equations of motion, which is common to both the *Entwurf* theory and the final theory. Perhaps this also helped to convince Einstein of his early misconceptions about the ease with which a reduction in the number of gravitational potentials could be achieved.

Thus, in 1912 and 1913 Einstein found himself driven to a single viewpoint by the interweaving of conclusions from quite disparate sources: in both weak and in static fields the number of gravitational potentials effectively reduces from ten to one in the same way. This agreement underpinned his confidence in equation (11) as the weak-field equations and in metric (1) as the metric of static fields and helped make his passage to his final field equations such a difficult one.

The account being developed here is supported by comments Einstein made late in 1915, after he returned to seek generally covariant field equations and began to realize his earlier mistakes. "The difficulty was not finding generally covariant equations for the $g_{\mu\nu}$," he wrote Hilbert in mid-November, "for this worked out easily with the help of the Riemann tensor. But it was difficult to recognize that these equations formed a generalization and, indeed, a simple and natural generalization of Newton's law."[33] A month later, after all the difficulties were finally resolved, Einstein wrote to Besso in a more buoyant mood:[34]

32. A very similar argument to this appears in Einstein, Infeld, and Hoffmann (ref. 37), 72–73.

33. Einstein to Hilbert, post-marked 18 Nov 1915, EA 13 091.

34. Einstein to Besso, 21 Dec 1915 (ref. 27), 61. See also Einstein to Besso, 10 Dec 1915, ibid., 59–60, for a briefer statement of surprise at the variability of $g_{11} - g_{33}$ in the weak-field case.

> The most delightful is the agreement of the perihelion motion and general covariance, the most remarkable however is the fact that Newton's theory *of the field* is already incorrect in equations of the 1st order (appearance of $g_{11}-g_{33}$). Only the fact that $g_{11}-g_{33}$ do not appear in the equations of motion effects the simplicity of Newton's theory.

Max Born, who was at this time in Berlin and in close contact with Einstein, surely had it on the best authority when he wrote in 1916:[35]

> What is remarkable about this [weak field] is that the g_{23}, g_{31} $\cdots$ in no way come out to be zero, so that there is more than one gravitational potential already in the first approximation; Einstein had first supposed the opposite and was forced into detours and incorrect assumptions before he found that his supposition was not confirmed.

The derivation of the Entwurf field equations

The awkward episode of his 1912 field equation in his earlier theory of static fields seems to have convinced Einstein of the necessity of ensuring from the very beginning that any new field equation satisfy the conservations laws. This means that one should be able to construct a gravitational field stress tensor, or a stress-energy tensor in the four dimensional case, using the field equations. This recognition provides the key to the understanding of some of the pages in the Zürich notebook and leads us directly to the derivation of the *Entwurf* field equations.

The law of conservation of energy-momentum, written as the vanishing of the covariant divergence of the stress-energy tensor $\Theta_{\mu\nu}$, takes the form

$$\sum_{\mu\nu}\frac{\partial}{\partial x_\nu}\left(\sqrt{-g}\,g_{\sigma\mu}\Theta_{\mu\nu}\right)-\frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\,\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\Theta_{\mu\nu}=0 \qquad (12)$$

in equation (10) of Einstein's part of the *Entwurf* paper. The second term of this expression can be interpreted as the rate of transfer of energy-momentum out of the gravitational field. Thus, by analogy with the first term in equation (12), it should be possible to set it equal to the coordinate divergence of a tensor density corresponding to the gravitational field stress-energy tensor $\theta_{\mu\nu}$. If we write this equation and then, using equation (2), substitute the gravitation tensor $\Gamma_{\mu\nu}$ for the stress-energy tensor $\Theta_{\mu\nu}$, we then recover the equation

35. M. Born, "Einsteins Theorie der Gravitation und der allgemeinen Relativität," *Physikalische Zeitschrift, 17* (1916), 51–59, on 58.

$$-\sum_{\mu\nu}\frac{\partial}{\partial x_\nu}\left(\sqrt{-g}\,g_{\sigma\mu}\kappa\theta_{\mu\nu}\right) = \frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\Gamma_{\mu\nu}\,. \tag{13}$$

This equation contains only the metric tensor and its derivatives and thus must be an identity. If we assume that the gravitation tensor has the form given in equation (3), then we can follow Einstein in taking the readily constructed

$$\sum_{\mu\nu\alpha\beta\tau\rho}\frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\left[\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} - \frac{1}{2}g_{\beta\sigma}\gamma_{\tau\rho}\frac{\partial g_{\mu\nu}}{\partial x_\tau}\frac{\partial \gamma_{\mu\nu}}{\partial x_\rho}\right]\right)$$
$$= \sum_{\mu\nu\alpha\beta}\sqrt{-g}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) \tag{14}$$

as approximating equation (13), where the equality in equation (14) is understood to hold only for quantities in the second order.

The fact that equation (13) should be an identity amounts to a simple test of whether the field equations in question satisfy the conservation equations and, conversely, provides a simple method of constructing the gravitational field stress-energy tensor. In the weak-field case considered here, we can read off an approximate expression for this tensor from the left-hand side of equation (14), by comparison with (13). On pages 19–21 of his notebook, immediately following the demonstration of how the harmonic coordinate condition reduces the Ricci tensor to the form (3), Einstein probed the relationship between his new field equations and the conservation laws by using exactly this method in the weak-field case. Further on pages 24–25, just after the construction of the gravitation tensor $\mathfrak{T}^{x}_{il}$, Einstein confirmed that the expression that he wrote as

$$\frac{d}{dx_i}\left(\gamma_{i\epsilon}\frac{\partial g_{\alpha\beta}}{\partial x_\epsilon}\frac{\partial \gamma_{\alpha\beta}}{\partial x_\sigma}\right) - \frac{1}{2}\frac{\partial}{\partial x_\sigma}\left(\gamma_{i\epsilon}\frac{\partial g_{\alpha\beta}}{\partial x_\epsilon}\frac{\partial \gamma_{\alpha\beta}}{\partial x_i}\right) \tag{15}$$

vanishes in Minkowski spacetime as viewed from Cartesian coordinates rotating at uniform speed. With summation over repeated indices, this expression is equivalent to the coordinate divergence of the gravitation field stress-energy tensor recoverable from (14), although the $\sqrt{-g}$ factor is missing and the limitation to weak fields no longer seems to apply. Since this tensor is derived from a gravitation tensor of the form (3), it seems that Einstein expected the gravitation tensor to have this form in the rotating-coordinate case considered here. Presumably his confirmation of the vanishing of this expression (15), which corresponds to the satisfaction of the conservation laws, would have confirmed for Einstein the correctness of this expectation.

We can now step directly to the method of deriving the field equations used in the *Entwurf* paper, which amounts to a simple inversion of the method used to construct equation (13). As Einstein showed in section 5 of his part of the paper, once an identity has been decided upon to stand for equation (13), then one reads the gravitation tensor and gravitational field stress-energy tensor directly from it. The derivation of this identity,

$$\sum_{\alpha\beta\tau\rho}\frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\frac{\partial\gamma_{\tau\rho}}{\partial x_\beta}\frac{\partial g_{\tau\rho}}{\partial x_\sigma}\right)-\frac{1}{2}\sum_{\alpha\beta\tau\rho}\frac{\partial}{\partial x_\sigma}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\frac{\partial\gamma_{\tau\rho}}{\partial x_\alpha}\frac{\partial g_{\tau\rho}}{\partial x_\beta}\right)$$

$$=\sum_{\mu\nu}\sqrt{-g}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\left\{\sum_{\alpha\beta}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\sqrt{-g}\frac{\partial\gamma_{\mu\nu}}{\partial x_\beta}\right)\right. \tag{16}$$

$$-\sum_{\alpha\beta\tau\rho}\gamma_{\alpha\beta}g_{\tau\rho}\frac{\partial\gamma_{\mu\tau}}{\partial x_\alpha}\frac{\partial\gamma_{\nu\rho}}{\partial x_\beta}+\frac{1}{2}\sum_{\alpha\beta\tau\rho}\gamma_{\alpha\mu}\gamma_{\beta\nu}\frac{\partial g_{\tau\rho}}{\partial x_\alpha}\frac{\partial\gamma_{\tau\rho}}{\partial x_\beta}$$

$$\left.-\frac{1}{4}\sum_{\alpha\beta\tau\rho}\gamma_{\mu\nu}\gamma_{\alpha\beta}\frac{\partial g_{\tau\rho}}{\partial x_\alpha}\frac{\partial\gamma_{\tau\rho}}{\partial x_\beta}\right\},$$

is given by Grossmann in his section of the paper. It amounts to a generalization of equation (14) from the weak-field case to the general case. Equation (14) was constructed originally by expanding the terms on its right-hand side and retaining only quantities of second order to yield the left-hand side. The bulk of Grossmann's derivation of identity (16) is devoted to making this process exact. He took the terms of third order, which were dropped in constructing equation (14), and reworked and redistributed them until the identity had the form required by equation (13). This yielded identity (16) directly.

For our purposes, the important point is that Einstein introduced these identities as "eindeutig bestimmt," "uniquely determined."[36] This, of course, suggests that he believed the *Entwurf* field equations to be uniquely determined, a belief that he stated explicitly a little later in 1913.[37] This conclusion seems to have put an end to his search for generally covariant field equations, so it is of great interest to us here. Neither Einstein nor Grossmann give any proof of their crucial result, the uniqueness of identity (16). On the basis of equations (13) and (3), Einstein required the identity to have the form

36. Einstein and Grossmann (ref. 1), 237.

37. Einstein, "Physikalische Grundlagen einer Gravitationstheorie," Naturforschende Gesellschaft, Zürich, *Vierteljahrsschrift*, *58* (1913), 284–290 (lecture of 9 Sep 1913), 289.

"sum of differential quotients

$$= \frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\,\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\left\{\sum_{\alpha\beta}\frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right)\right. \tag{17}$$

$$\left. + \text{ further terms which fall away in the first approximation}\right\} ."$$

It is necessary to spell out the conditions which this identity had to satisfy more clearly; otherwise there is no possibility of developing a uniqueness proof. Most naturally, we would require the gravitational field stress-energy tensor to be a sum of terms quadratic in the first derivatives of the metric tensor. From comparison with equation (13), we see that this requirement specifies the form of the left-hand side of equation (17), which is just the usual divergence of this tensor. Following the form of Einstein's final result, we can also require that the "further terms" on the right-hand side can be constructed out of the metric tensor and its first derivatives only.

It is a little surprising to find that these natural, even somewhat restrictive, conditions still do not specify the identity uniquely. It is possible to add further terms to equation (16), which would still leave the equation an identity and do not violate any of the above conditions.[38] This suggests that Einstein was mistaken in his claim that the identity (16) was unique and thus mistaken in his belief in the uniqueness of the *Entwurf* field equations. Before we convict Einstein of this mistake—and perhaps also Grossmann, although the uniqueness claim does not appear in his part of the paper—we should consider the possibility that they placed an additional constraint on the form of the left-hand side of the identity.

In identity (16) the left-hand side contains only quantities built up out of the derivatives of the metric tensor in which the $\mu\nu$ indices of

38. For example, consider the following identity:

$$\Big(2\sqrt{-g}\; g_{\tau\tau'}\, g^{\tau\alpha}_{,\sigma}\, g^{\tau'\beta}_{,\beta} - \delta^\alpha_\sigma\sqrt{-g}\; g_{\tau\tau'}\, g^{\tau\alpha'}_{,\alpha'}\, g^{\tau'\beta}_{,\beta} - 2\sqrt{-g}\; g_{\tau\beta}\, g^{\tau\rho}_{,\sigma}\, g^{\beta\alpha}_{,\rho}$$
$$+\delta^\alpha_\sigma\sqrt{-g}\; g_{\alpha'\beta}\, g^{\alpha'\rho}_{,\tau}\, g^{\beta\tau}_{,\rho}\Big)_{,\alpha} \equiv g_{\mu\nu,\sigma}\sqrt{-g}\; g^{\mu\mu'} g^{\nu\nu'}\Bigg[\frac{1}{\sqrt{-g}}\Big((\sqrt{-g}\; g_{\alpha\mu'})_{,\sigma'}\, g^{\alpha\sigma'}_{,\nu'}$$
$$+ (\sqrt{-g}\; g_{\alpha\nu'})_{,\sigma'}\, g^{\alpha\sigma'}_{,\mu'} - (\sqrt{-g}\; g_{\tau\mu'})_{,\nu'}\, g^{\tau\alpha}_{,\alpha} - (\sqrt{-g}\; g_{\tau\nu'})_{,\mu'}\, g^{\tau\alpha}_{,\alpha}\Big)$$
$$+\frac{1}{2} g_{\mu'\nu'}\Big(g_{\alpha\beta}\, g^{\alpha\rho}_{,\tau}\, g^{\beta\tau}_{,\rho} - g_{\tau\tau'}\, g^{\tau\alpha}_{,\alpha}\, g^{\tau'\beta}_{,\beta}\Big) + g_{\alpha\mu'} g^{\alpha\rho}_{,\tau}\, g_{\beta\nu'}\, g^{\beta\tau}_{,\rho} - g_{\tau\mu'}\, g^{\tau\alpha}_{,\alpha}\, g_{\tau'\nu'}\, g^{\tau'\beta}_{,\beta}\Bigg] .$$

This identity can be added to identity (16) to yield a new identity consistent with all the conditions stated so far. Note that the right-hand side of the identity written above vanishes to quantities in the second order of smallness. The discovery of such identities is by no means easy. This one was found by comparison of different gravitation tensors generated by Lagrangian methods.

these derivatives $\frac{\partial g_{\mu\nu}}{\partial x_\alpha}$ are always summed as $\sum_{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x_\alpha} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}$. Perhaps this was no accident. Einstein and Grossmann may have required that the terms on the left-hand side have this form only. They do not state this explicitly. However they may have tried to indicate it with their use of the term "sum of differential quotients." Their use of this term is sufficiently restricted in the *Entwurf* paper to allow it to refer to quantities that are not just the sum of differential quotients, but of differential quotients summed in the special way I have indicated here.

Why they would impose such a restriction is not entirely clear. The gravitational field stress-energy tensor derived in the weak-field case from equation (14) is built up out of terms of this form alone. Perhaps they wanted to retain this form in the *Entwurf* case, which was built up as a generalization of the weak-field case. Given the assumption that the gravitation field stress-energy tensor has to be made up out of terms quadratic in the derivatives of the metric tensor, we can add no further terms to the existing terms of the weak-field stress-energy tensor without interfering with the reduction to the weak-field case of (14). This would certainly restrict the summation to the way described above, for it would require the gravitational field stress-energy tensor to be exactly the one derived from equation (14).

In any case, if Einstein and Grossmann did use this additional constraint, then it can be proved that their identity (16) is unique. (The proof is deceptively difficult.) The *Entwurf* field equations would be uniquely determined. Thus, with the conclusion of his work on the *Entwurf* paper, Einstein believed that he had found that the naturally suggested generally covariant tensors were inadmissible as gravitation tensors since they failed to yield the correct Newtonian limit and, furthermore, that he had been able to derive the only acceptable field equations, which did not turn out to be generally covariant.

It is clear that the problem in the derivation rests on Einstein's misconception of the Newtonian limit. The general method is sound. The familiar contracted Bianchi identities can be written in the form of equation (13) and from them Einstein's final generally covariant field equations recovered. But Einstein ruled out consideration of this identity by requiring the right-hand side of the identity to have the form (17). This meant that he could only recover a gravitation tensor with the form (3) from his identities: the use of the Bianchi identities would yield the Einstein tensor as the gravitation tensor and it does not have this form. Moreover, Einstein's method could not even yield the field equations of his final general theory of relativity after they had been reduced by some coordinate condition. Einstein's method assumes that the weak-field equations, in appropriate coordinate systems, take the form (11), whereas his final theory yields the weak-field equations (11′)

in similar circumstances.

Finally, Einstein still admitted the possibility of generally covariant field equations if derivatives higher than the second in the metric tensor were allowed. Indeed, in 1914 he insisted that there must be a generally covariant generalization of the field equations if they have any physical content.[39] But this belief seems not to have had any practical effect on his work.

5. THE ARGUMENTS AGAINST GENERAL COVARIANCE

With the conclusion of the *Entwurf* paper, the problem of the field equations had altered radically. The question was no longer "What are the generally covariant field equations?" It had become "Why are there not any second order generally covariant field equations?" and "How does the limited covariance of the field equations fit with the requirement of the general relativity of motion?" Once Einstein took this approach, answers to the first question came fairly fast.

We can date the discovery of the first answer to the first question quite exactly—August 15, 1913—from Einstein's correspondence with Lorentz.[40] Einstein's analysis focussed on the law of conservation of energy-momentum, written as

$$\sum_{\nu} \frac{\partial}{\partial x_{\nu}} \left(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu} \right) = 0 \,, \tag{18}$$

where $\mathfrak{T}_{\sigma\nu}$ and $\mathfrak{t}_{\sigma\nu}$ are the mixed stress-energy tensor densities for matter and the gravitational field respectively (the ν index is contravariant). Einstein's argument has the general form of a *reductio ad absurdum.* If the theory were generally covariant, then we would expect the stress-energy tensor for the gravitational field to be generally covariant and share the same transformation properties as the usual stress-energy tensor since they both enter into equations such as (18) in the same way. But if this were the case, equation (18) could not be generally covariant. Indeed, a "closer consideration" shows that such an equation (18) could only be covariant under linear coordinate transformations.[41] This means that the conservation laws and, as a result, the

39. Einstein, "Prinzipielles zur verallgemeinerten Relativitätstheorie," *Physikalische Zeitschrift 15* (1914), 176–180 (received 24 Jan 1914), 177–178.

40. A. Einstein to H. A. Lorentz, 16 Aug 1913, EA 16 434. See also Einstein and Grossmann (ref. 1), 260–261; Einstein (ref. 31), 1257–1258, and (ref. 48), 178.

41. Einstein (ref. 48), 178. Einstein did not tell us what this "closer consideration" was. Perhaps he wrote $1/\sqrt{-g}$ times the left-hand side of equation (18) in terms of a covariant divergence in the following way:

$$1/\sqrt{-g}\left(\sqrt{-g}\,T^{\nu}_{\sigma} + \sqrt{-g}\,t^{\nu}_{\sigma}\right)_{,\nu} = \left(T^{\nu}_{\sigma} + t^{\nu}_{\sigma}\right)_{;\nu} + \frac{1}{2} g_{\mu\nu,\sigma}\left(T^{\mu\nu} + t^{\mu\nu}\right) ,$$