in similar circumstances.

Finally, Einstein still admitted the possibility of generally covariant field equations if derivatives higher than the second in the metric tensor were allowed. Indeed, in 1914 he insisted that there must be a generally covariant generalization of the field equations if they have any physical content.[39] But this belief seems not to have had any practical effect on his work.

5. THE ARGUMENTS AGAINST GENERAL COVARIANCE

With the conclusion of the *Entwurf* paper, the problem of the field equations had altered radically. The question was no longer "What are the generally covariant field equations?" It had become "Why are there not any second order generally covariant field equations?" and "How does the limited covariance of the field equations fit with the requirement of the general relativity of motion?" Once Einstein took this approach, answers to the first question came fairly fast.

We can date the discovery of the first answer to the first question quite exactly—August 15, 1913—from Einstein's correspondence with Lorentz.[40] Einstein's analysis focussed on the law of conservation of energy-momentum, written as

$$\sum_{\nu} \frac{\partial}{\partial x_{\nu}} \left(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu} \right) = 0 , \tag{18}$$

where $\mathfrak{T}_{\sigma\nu}$ and $\mathfrak{t}_{\sigma\nu}$ are the mixed stress-energy tensor densities for matter and the gravitational field respectively (the ν index is contravariant). Einstein's argument has the general form of a *reductio ad absurdum.* If the theory were generally covariant, then we would expect the stress-energy tensor for the gravitational field to be generally covariant and share the same transformation properties as the usual stress-energy tensor since they both enter into equations such as (18) in the same way. But if this were the case, equation (18) could not be generally covariant. Indeed, a "closer consideration" shows that such an equation (18) could only be covariant under linear coordinate transformations.[41] This means that the conservation laws and, as a result, the

39. Einstein, "Prinzipielles zur verallgemeinerten Relativitätstheorie," *Physikalische Zeitschrift 15* (1914), 176–180 (received 24 Jan 1914), 177–178.

40. A. Einstein to H. A. Lorentz, 16 Aug 1913, EA 16 434. See also Einstein and Grossmann (ref. 1), 260–261; Einstein (ref. 31), 1257–1258, and (ref. 48), 178.

41. Einstein (ref. 48), 178. Einstein did not tell us what this "closer consideration" was. Perhaps he wrote $1/\sqrt{-g}$ times the left-hand side of equation (18) in terms of a covariant divergence in the following way:

$$1/\sqrt{-g}\left(\sqrt{-g}\,T^{\nu}_{\sigma} + \sqrt{-g}\,t^{\nu}_{\sigma}\right)_{,\nu} = \left(T^{\nu}_{\sigma} + t^{\nu}_{\sigma}\right)_{;\nu} + \frac{1}{2} g_{\mu\nu,\sigma}\left(T^{\mu\nu} + t^{\mu\nu}\right) ,$$

theory as a whole, can only hold in coordinate systems related by linear coordinate transformations, which contradicts the assumed general covariance.

Einstein was very pleased and, perhaps, even somewhat relieved with the discovery of this argument. In his letter to Lorentz he confided: "Only now does the theory please me, after this ugly dark spot seems to have been removed." We can understand Einstein's satisfaction, for there is a pleasing coherence in the way that the conservation laws first circumscribe powerfully the range of admissible field equations, as he found in the derivation of the *Entwurf* field equations, and then guarantee that the theory cannot be generally covariant. After Einstein felt that he had stronger reasons for rejecting the admissibility of generally covariant field equations, he was still pleased to note that the stronger restriction of the covariance of the theory to *linear* coordinate transformations should follow from the conservation laws. He wrote Ehrenfest: "What can be more beautiful than that necessary specialization flowing from the conservation. laws."[42]

Of course what Einstein had not allowed for in his argument is that a generally covariant theory, with generally covariant field equations, might have a stress-energy tensor for the gravitational field which is itself not a generally covariant tensor, without compromising the general covariance of the theory. This is the case in the final general theory of relativity. That Einstein should miss this point is by no means a trivial oversight. The basic nature of the theory seems to demand that any physically meaningful quantity be represented by a generally covariant tensor. That this is not the case for the gravitational field stress-energy tensor of the final theory was to be a source of some confusion and is now explained in terms of the impossibility of localizing gravitational field energy and momentum.

Certainly we could not expect Einstein to anticipate this at a time when he still believed that there were no second order, generally covariant field equations. There was, however, a second error in Einstein's use of this consideration. Einstein had concluded from it that the covariance of all versions of his new theory would be limited to linear transformations. This included the *Entwurf* theory. But, as we have seen, the restriction to linear coordinate transformations depended on the general covariance of the stress-energy tensor for the gravitational field and this tensor in the *Entwurf* theory was not generally covariant. So the argument from the conservation law did not entail a

All terms on the right-hand side are generally covariant tensors with the exception of $g_{\mu\nu,\sigma}$ which is a tensor under linear coordinate transformations only. Thus the left-hand side can only be covariant under linear coordinate transformations as well.

42. Einstein to Ehrenfest, 1913, EA 9 342.

limitation of the covariance of the *Entwurf* theory to linear coordinate transformations. Einstein himself pointed out this error early in 1914 in a footnote to the paper in which he began concentrated study of the covariance properties of the *Entwurf* theory.[43] This was an important point, since Einstein came to regard it as essential that the *Entwurf* theory be covariant under more than just linear coordinate transformations if it was to realize any extension of the principle of relativity.

The argument from the conservation law then seems to have dropped from sight. It had already been eclipsed by what seemed to be a stronger argument against generally covariant field equations, the "hole" argument, discovered sometime late in 1913 and 1914.[44] The argument is intended to show that if the field equations are generally covariant, then a given stress-energy tensor cannot uniquely determine the gravitational field through the field equations. The first three versions of the argument are virtually identical. I quote the second version:

> *If the reference system is chosen quite arbitrarily, then in general the $g_{\mu\nu}$ cannot be completely determined by the $\mathfrak{T}_{\sigma\nu}$* [stress-energy tensor density]. For, think of the $\mathfrak{T}_{\sigma\nu}$ and $g_{\mu\nu}$ as given everywhere and let all $\mathfrak{T}_{\sigma\nu}$ vanish in a region of Φ of four dimensional space. I can now introduce a new reference system, which coincides completely with the original outside Φ, but is different to it inside Φ (without violation of continuity). One now relates everything to this new reference system, in which matter is represented by $\mathfrak{T}'_{\sigma\nu}$ and the gravitational field by $g'_{\mu\nu}$. Then it is certainly true that
>
> $$\mathfrak{T}'_{\sigma\nu} = \mathfrak{T}_{\sigma\nu}$$
>
> everywhere, but against this the equations
>
> $$g'_{\mu\nu} = g_{\mu\nu}$$
>
> will definitely not all be satisfied inside Φ. The assertion follows from this.
>
> If one wants a complete determination of the $g_{\mu\nu}$ (gravitational field) by the $\mathfrak{T}_{\sigma\nu}$ (matter) to be possible, then this can only be achieved by a limitation on the choice of reference systems.

43. Einstein and Grossmann, "Kovarianzeigenschaften der Feldgleichungen der auf die verallgemeinerte Relativitätstheorie gegründeten Gravitationstheorie," *Zeitschrift für Mathematik und Physik*, *63* (1914), 215–225, on 218.

44. Ordered by dates of publication, Einstein and Grossmann (ref. 1), 260–261; Einstein (ref. 39), 178; Einstein and Grossmann (ref. 43), 217–218; Einstein (ref. 26), 1066–1067. The argument does not appear in the body of the published text of Einstein's address to the Congress of German Natural Scientists and Physicians in 1913, although it appears in a footnote to the printed text (ref. 38), 1247; it occurs in a letter to L. Hopf of 2 Nov 1913 (EA 13 290) and in the addendum to the journal printing of the *Entwurf* paper and not in the earlier separatum.

Einstein claimed to have shown with this argument that a single stress-energy tensor can determine two different gravitational fields, if the field equations are generally covariant. Einstein's argument seems to rest on a simple mistake: $g'_{\mu\nu}$ and $g_{\mu\nu}$ do not represent different gravitational fields. Rather they represent the same gravitational field, but as it appears in two different coordinate systems. All Einstein seems to have shown is that a given gravitational field will look different if viewed from different coordinate systems. On this basis, there seems no reason to doubt that the given stress-energy tensor does specify a unique gravitational field. This mistake is a trivial one and it has become customary in accounts of this argument to convict Einstein of making it.[45] Grossmann must then also have made the same mistake, for he was still collaborating with Einstein at this time and even coauthored one of the papers in which the argument appears.

That both Einstein and Grossmann could repeatedly make this same trivial mistake on such an important question is highly implausible, especially if we recall that Einstein was quite comfortable with the notion of applying coordinate conditions to generally covariant gravitation tensors prior to the completion of the *Entwurf* paper. Moreover, there is unequivocal evidence in both of the papers, in which the third and fourth versions of the "hole" argument appeared, that Einstein and Grossmann recognized that a change in coordinate system did not produce a new field, even though the components of the metric tensor may change.[46] In both papers, an arbitrary infinitesimal change in the ten components of the metric tensor $g_{\mu\nu}$ is introduced. This is broken up into two parts, the first of which corresponds to a change in the gravitational field between "adapted" coordinates, the second to a change "that can be produced through mere variation of the coordinate system without a change of the gravitational field.... A variation of this kind is determined by four functions (variations of the coordinates), which are are independent of one another."[47] This shows a clear recognition of the fact that a change in the coordinate system does not alter the gravitational field, although the components of the metric tensor will change, and that this arbitrariness in the representation of the field is associated with four independent conditions.

Fortunately, it is possible to give a quite different account of the content and import of the "hole" argument, as it appears in its fourth version, which does not convict Einstein of a trivial mistake, and I will argue that this interpretation can also be used on the earlier three versions. The essential part of the text of this fourth version reads as

45. See ref. 18 and ref. 19.
46. Einstein and Grossmann (ref. 43), 223; Einstein (ref. 26), 1071.
47. Einstein (ref. 26), 1071–1072.

follows:

> We consider a finite region of the continuum Σ, in which no material process takes place. Physical happenings in Σ are then fully determined, if the quantities $g_{\mu\nu}$ are given as functions of the x_ν in relation to the coordinate system K used for description. The totality of these functions will be symbolically denoted by $G(x)$.
>
> Let a new coordinate system K' be introduced, which coincides with K outside Σ, but deviates from it inside Σ in such a way that the $g'_{\mu\nu}$ related to the K' are continuous everywhere like the $g_{\mu\nu}$ (together with their derivatives). We denote the totality of the $g'_{\mu\nu}$ symbolically with $G'(x')$. $G'(x')$ and $G(x)$ describe the same gravitational field. In the functions $g'_{\mu\nu}$ we replace the coordinates x'_ν with the coordinates x_ν, i.e., we form $G'(x)$. Then, likewise, $G'(x)$ describes a gravitational field with respect to K, which however does not correspond with the real (or originally given) gravitational field.
>
> We now assume that the differential equations of the gravitational field are generally covariant. Then they are satisfied by $G'(x')$ (relative to K') if they are satisfied by $G(x)$ relative to K. Then they are also satisfied by $G'(x)$ relative to K. Then relative to K there exist the solutions $G(x)$ and $G'(x)$, which are different from one another, in spite of the fact that both solutions coincide in the boundary region, i.e., *happenings in the gravitational field cannot be uniquely determined by generally covariant differential equations for the gravitational field.*

This version of the argument is identical to the three earlier versions, with the exception of the addition of a new and crucial step at the end. This step involves the construction of a new gravitational field that is also a solution of the field equations with the same stress-energy tensor and in the same coordinate system as the original field. It makes clear that the introduction of the alternate coordinate representation $G'(x')$ of the original field is only a device to enable construction of the new field. There is clearly no confusion over whether $G'(x')$ represents a new field. Einstein wrote: "$G'(x')$ and $G(x)$ describe the same gravitational field."

The way in which Einstein constructed this new field in the argument does bear some elucidation since Einstein's account of it is quite brief. The construction proceeds as follows. Consider a particular point of the spacetime manifold, called P_1 for convenience, in Σ. It will have coordinates x_ν in K and x'_ν in K'. There will be another point P_2 in Σ whose coordinates *in* K' are numerically the same as x_ν. The gravitational field at this point P_2 in coordinate system K' will be described by the functions $G'(x_\nu)$. Now consider the new field that would arise at the original point P_1 if the functions describing the field at P_1 were not $G(x_\nu)$ but $G'(x_\nu)$. Clearly the fields described by the functions $G(x_\nu)$ and $G'(x_\nu)$ are related to the same coordinate system K since the arguments of both functions are the same numbers x_ν, the coordinates of P_1

in K. But equally clearly they cannot describe the same field since G and G' are not the same functions. If a new field is constructed in this way for all points in the spacetime manifold, then this new field will still satisfy the field equations. For we have done nothing to change the *mathematical* form of $G'(x'_\nu)$, which, of course, is a solution to the field equations, in constructing the new field. Rather, all that has been done occurs on the conceptual level. That is, we reassign the points in the spacetime manifold which are thought of as belonging to a given set of coordinates—specifically, the point P_1 in the manifold is now assigned to the coordinates x_ν in the coordinate system K'. Since the stress-energy tensor vanishes everywhere in Σ, its new components, generated by exactly the same method as the new field, will still vanish everywhere in Σ. That is, its components will agree everywhere with those of the original stress-energy tensor in K. Thus both the new field and the old field are solutions to the field equations with the same stress-energy tensor in the same coordinate system K—and Einstein's result is established.

The case of the three earlier versions of the argument still remains. Were it not for one crucial piece of evidence, it would be difficult to escape the conclusion that Einstein and Grossmann were presenting a different argument to the fourth version and one in which the trivial mistake outlined above is committed. That crucial piece of evidence is a footnote appended to the sentence ending "the equations $g'_{\mu\nu} = g_{\mu\nu}$ will definitely not all be satisfied inside Φ," in the second version of the argument quoted above:

> The equations are to be understood in such a way that each of the independent variables x'_ν on the left-hand side are to be given the same numerical values as the variables x_ν on the right-hand side.

In other words, Einstein required the inequality of $g'_{\mu\nu}$ and $g_{\mu\nu}$ to be read in a special way. In terms of points P_1 and P_2 defined earlier, the $g'_{\mu\nu}$ at P_2 are unequal to the $g_{\mu\nu}$ at P_1. This, of course, is the inequality crucial to the fourth version of the argument. If all Einstein were saying was that the different coordinate representations of the original field $g_{\mu\nu}$ and $g'_{\mu\nu}$ were actually different fields, then there would have been no reason to specify that the inequality be read in this special manner.

This suggests that all four versions of the argument were understood by Einstein to have the same content as the fourth and that his greatest mistake was only to present the first three versions in too compact a form to be readily understood. Presumably the crucial stipulation on how the inequality of the $g_{\mu\nu}$ and $g'_{\mu\nu}$ was to be read was obvious to Einstein, for it was appended only as an apparent afterthought to the second version of the argument in the footnote quoted. (It could equally have been added to the first and third versions and thus made their content clearer as well.) Perhaps if we were all Einsteins then such subtleties would be equally clear to us too!

Stachel has pointed out that, in effect, what Einstein did in the final step of the argument is to generate a new field from the old one by means of a point transformation.[48] Specifically, under a point transformation from P_2 to P_1, the image field represented in the image of coordinate system K' amounts to the field with coordinate representation $G'(x)$. Further, Stachel has argued that, with the discovery of his generally covariant field equations in 1915, Einstein was able to draw a very significant physical conclusion about the relationship between the spacetime manifold and the metric field from the machinery of the "hole" argument. This was that the individual points of the spacetime manifold have no independent individuality and can only be distinguished with reference to the metric field (or perhaps some other material phenomena) in spacetime. Thus it is the final step of the "hole" argument that is erroneous.

Within this understanding, it is impossible to drag the metric field away from a physical point in empty spacetime and leave that physical point behind. For the physical individuation of the point only has meaning in terms of the metric field at that point. Or, in the terms Einstein used in the fourth version of the argument, it makes no sense to remove one field, $G(x)$, leave behind the bare spacetime manifold, as represented by the coordinate system K, and then construct a new field, $G'(x)$, on this bare manifold. For this presupposes the concept of a spacetime manifold, replete with points that have an existence independent of the metric field. Take away the metric field and one takes away the spacetime points with it.[49]

This account is derived from—and indeed explains—Einstein's comments about the "hole" argument made late in 1915 and early 1916, after the discovery of the final generally covariant field equations. He wrote Besso:[50]

> Everything was correct in the hole consideration up to the last conclusion. There is no physical content in two different solutions $G(x)$ and $G'(x)$ existing with respect to the *same* coordinate system K. To imagine two solutions simultaneously in the same manifold has no meaning and the system K has no physical reality. In the place of the hole consideration we have the following. *Reality* is nothing but the totality of space-time point coincidences. If, for example, physical happenings

48. Stachel (ref. 3).

49. Einstein later put great stress on this inseparability of metric and manifold. See his *Relativity: The special and the general theory* (London, 1977), 155.

50. Einstein to Besso, 3 Jan 1916 (ref. 27), 63–64. Cf. Einstein to Ehrenfest, 26 Dec 1915, EA 9 363.

> could be built up out of the motion of material points alone, then the meetings of the points, i.e., the points of intersection of their world lines, would be the only reality, i.e., in principle observable. Naturally these points of intersection remain unchanged in all transformations (and no new ones are added) only if certain uniqueness conditions are preserved. Therefore it is most natural to require of laws that they determine no more than the totality of timespace coincidences. Following what has been said before, this is already achieved with generally covariant equations.

One of the important outcomes of Einstein's experience with the "hole" argument was the point coincidence argument for the need of generally covariant equations, which is sketched out in this letter to Besso. Einstein came to use this argument to good effect in his expositions of the general theory of relativity.[51]

6. COVARIANCE PROPERTIES OF THE ENTWURF FIELD EQUATIONS

Now satisfied that there were good reasons to give up a search for a generally covariant theory, Einstein could devote his attention to the task of elucidating his *Entwurf* theory and, in particular, of determining the significance of its limited covariance. He began this work before his move to Berlin, while still collaborating with Grossmann, and the first product of their labor appeared early in 1914. In the introduction to this paper, they stressed that the field equations must be covariant under nonlinear coordinate transformations as well as linear if the theory was to contain an extension of the principle of relativity and satisfy the requirements of the principle of equivalence. They summarized the achievements of the paper:[52]

> In the following it will be proved that the gravitation equations set up by us have that degree of general covariance which is conceivable under the condition that the fundamental tensor $g_{\mu\nu}$ should be completely determined by the gravitation equations; in particular, it turns out that the gravitation equations are covariant under acceleration transformations (i.e., non-linear transformations) of many different kinds.

The "condition that the fundamental tensor $g_{\mu\nu}$ should be completely determined by the gravitation equations" refers, of course, to the "hole" argument. This passage therefore asserted that the *Entwurf* field equations have the maximum covariance consistent with the considerations

51. For example, Einstein, "Die Grundlage der allgemeinen Relativitätstheorie," *AP*, *49* (1916), 769–822 (received 20 Mar 1916), 776–777.

52. Einstein and Grossmann (ref. 43), 216.

of the "hole" argument. They proved this assertion simply and elegantly. However, there is no further detailed discussion of the second assertion, that this allowed covariance embraces a wide range of acceleration transformations, or any demonstration that some of these correspond to cases of special physical interest in the context of a generalized principle of relativity or the principle of equivalence. This is a curious and, as it turned out, serious omission and one that was maintained in Einstein's generalization of the work in this paper late in 1914.

The paper continued with a brief statement of the equations of the *Entwurf* theory and of the "hole" argument itself. Einstein and Grossmann then took the field equations, written in the compact form

$$\sum_{\alpha\beta\mu} \frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta} g_{\sigma\mu} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) = \kappa\left(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu}\right) , \tag{19}$$

formed its coordinate divergence, and then applied the conservation law in the form of equation (18). This resulted in the condition

$$B_\sigma = \sum_{\alpha\beta\mu\nu} \frac{\partial^2}{\partial x_\nu \partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta} g_{\sigma\mu} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) = 0 , \tag{20}$$

which would clearly have to be satisfied in any coordinate system in which the *Entwurf* field equations held. They decided to call such coordinate systems "adapted."

The remainder of the paper was devoted to showing that this necessary condition is also a sufficient one. That is, if condition (20) was still satisfied after a coordinate transformation, then the field equations would still hold in the new coordinate system. In their proof, Einstein and Grossmann introduced mathematical techniques which would become of great importance to the development of the theory. They found a variational formulation of the field equations and studied their covariance properties by examining the behavior of the variational integral under an infinitesimal coordinate transformation. I pass over the details of their arguments now, for they are fully subsumed by the more general apparatus of Einstein's first major paper on the theory after his move to Berlin in April 1914. With that Einstein could guarantee that his field equations had the maximum covariance permitted by the "hole" argument.

We can approach the condition (20) in terms of the problem of coordinate conditions discussed earlier. Einstein had made clear in a paper written earlier in 1914 that he believed that some generally covariant set of equations must correspond to the *Entwurf* equations.[53]

53. Einstein (ref. 39), 178.

The condition (20) would then be the coordinate condition that would have to be applied to these equations in order to recover their *Entwurf* specialization.[54] Of course by this time Einstein had become convinced that the field equations in their generally covariant form were physically uninteresting as field equations. So there would have been little to gain from finding them.

The results of his and Grossmann's paper seemed at last to have reconciled Einstein to the limited covariance of his theory. He described these results to his friend Besso in a letter of March 1914, in which he claimed that the justified transformations included rotations transformations:[55]

> Now I am completely satisfied and no longer doubt the correctness of the whole system, whether the observation of the solar eclipse work out or not. The sense [*Vernunft*] of the matter is too evident.

Einstein's move to Berlin in April 1914 marked the end of his collaboration with Grossmann. Fortunately, by this time Einstein no longer seems to have needed Grossmann's mathematical guidance. By October 1914, he had completed a lengthy summary article on his new theory, whose form and detailed nature suggest that Einstein felt his theory had reached its final form. The article contained a review of the methods of tensor calculus used in the theory and, flexing his newfound mathematical muscles, Einstein could even promise to give new and simpler derivations of the basic laws of the "absolute differential calculus."[56] Of great importance for us was the fact that Einstein had taken the new mathematical techniques of his last paper with Grossmann, generalized them and found in them a quite new derivation of the field equations.

Einstein began his new derivation and treatment of the covariance properties of the field equations by introducing an undetermined action H (for "Hamiltonian"?), which was to be some function of the metric tensor $g^{\mu\nu}$ and its first derivatives $g^{\mu\nu}_{\sigma}$.[57] From this the integral

$$J = \int H\sqrt{-g}\,d\tau \tag{21}$$

was formed, where $d\tau$ is an infinitesimally small element of spacetime.

54. Einstein treated condition (20) in the same way as the coordinate condition (10) in his letter to Paul Hertz (ref. 29).

55. Einstein to Besso, Mar 1914 (ref. 27), 52–53.

56. Einstein (ref. 26), 1030.

57. Ibid., 1066–77. In this paper, Einstein reintroduced the representation of contravariant components of a tensor by raised indices and covariant components by lowered indices. This convention had been used by Ricci and Levi-Civitá. Grossmann (ref. 1), 246, described how he and Einstein decided not to use it then because it was too complicated in certain cases. The term $g^{\mu\nu}_{\sigma}$ signifies $\partial g^{\mu\nu}/\partial x_{\sigma}$.

The $g^{\mu\nu}$ were varied infinitesimally in such a way that their variation $\delta g^{\mu\nu}$ disappeared on the boundary of the region of spacetime of the integration.

The variation produced in J can be rewritten as

$$\delta J = \int \sum_{\mu\nu} \mathfrak{E}_{\mu\nu} \delta g^{\mu\nu} d\tau \ , \tag{22}$$

and the new quantity,

$$\mathfrak{E}_{\mu\nu} = \frac{\partial H\sqrt{-g}}{\partial g^{\mu\nu}} - \sum_{\sigma} \frac{\partial}{\partial x_\sigma} \left(\frac{\partial H\sqrt{-g}}{\partial g_\sigma^{\mu\nu}} \right) , \tag{23}$$

was designated as the tensor density derived from the gravitation tensor for the field equations (2). The covariance properties of the field equations that resulted were determined by examining the behaviors of J under infinitessimal coordinate transformations. Introducing such a transformation Δ, Einstein found that

$$\frac{1}{2}\Delta H = \sum_{\mu\nu\sigma\alpha} g^{\nu\alpha} \frac{\partial H}{\partial g_\sigma^{\mu\nu}} \frac{\partial^2 \Delta x_\mu}{\partial x_\sigma \partial x_\alpha} , \tag{24}$$

given that H was a function of $g^{\mu\nu}$ and $g_\sigma^{\mu\nu}$ alone and making the assumption that H was invariant under linear transformations, which made it possible to neglect all terms in $\frac{\partial \Delta x_\mu}{\partial x_\sigma}$ in the general expression for ΔH. From this assumption, it followed that

$$\frac{1}{2}\Delta J = \int d\tau \sum_{\mu} (\Delta x_\mu B_\mu) + F , \tag{25}$$

where

$$B_\mu = \sum_{\alpha\sigma\nu} \frac{\partial^2}{\partial x_\sigma \partial x_\alpha} \left(g^{\nu\alpha} \frac{\partial H\sqrt{-g}}{\partial g_\sigma^{\mu\nu}} \right) , \tag{26}$$

and F is a surface integral term that would vanish in case Δx_μ and $\frac{\partial \Delta x_\mu}{\partial x_\sigma}$ vanished on the boundary of the region of integration.

Einstein could then define "adapted" coordinates for a given field. He considered a series of infinitesimally separated coordinate systems K, K', K'', $\cdots$, whose values agreed on the boundary of the region of integration in such a way that if Δ represented the coordinate transformation between two adjacent systems, then both Δx_μ and $\frac{\partial \Delta x_\mu}{\partial x_\sigma}$ disappear on the boundary: F vanishes for Δ. Therefore we have

$$\frac{1}{2}\Delta J = \sum_{\mu} \int d\tau \Delta x_\mu B_\mu \ . \tag{27}$$

Einstein defined the adapted coordinate system to be that one for which J is an extremum: in adapted coordinate systems,

$$B_\mu = 0 \ . \tag{28}$$

Einstein proceeded to demonstrate his crucial result that

$$\Delta(\delta J) = 0 \tag{29}$$

provided that Δ relates adapted coordinate systems, or, using Einstein's term, is a "justified" coordinate transformation. In other words, δJ is a scalar under justified coordinate transformations. It follows directly from equation (22) that the gravitation tensor $1/\sqrt{-g}\ \mathfrak{G}_{\mu\nu}$ is a tensor under justified coordinate transformations.

These results have been derived so far with a largely undetermined H. If at the beginning H had been set as

$$H = \frac{1}{4}\sum_{\alpha\beta\tau\rho} g^{\alpha\beta}\frac{\partial g_{\tau\rho}}{\partial x_\alpha}\frac{\partial g^{\tau\rho}}{\partial x_\beta}\ , \tag{30}$$

then these calculations would have corresponded to those of Einstein and Grossmann's earlier paper, in which they showed that the *Entwurf* field equations are covariant under justified coordinate transformations. The gravitation tensor resulting from this choice of H is the *Entwurf* gravitation tensor, as Einstein and Grossmann showed, and the condition (28) takes on the form of condition (20). Further, we can now see that the basis of Einstein and Grossmann's claim that the *Entwurf* field equations have the maximum covariance allowed by the "hole" argument and can also note that this applies to the generalized field equations of Einstein's Berlin paper as well.

The "hole" argument was built on the fact that generally covariant field equations in a given region of spacetime hold in any two coordinate systems whose values agree on the boundary of that region. Einstein's definition of an adapted coordinate system requires the selection of one of all those coordinate systems in a given region of spacetime, whose values agree on the boundary, by means of the condition that ΔJ in equation (27) be an extremum.[58] Such a restriction seems to be the minimum that the "hole" argument requires and this is the full extent of the limitation of the covariance of Einstein's field equations.[59]

In the Berlin paper, Einstein no longer felt that he had to stipulate the value of H in order to recover his *Entwurf* field equations. He

58. Cf. Einstein to Lorentz, 23 Jan 1915, EA 16 436.

59. The close connection between the devices employed in the "hole" argument and in the variational treatment of the field equations suggests that the "hole" argument may have occurred to Einstein as a result of early attempts to apply variational techniques to his field equations.

believed that he could derive equations (30) from the above general formulation of the field equations. To do this, he substituted the gravitation tensor of equation (23) into the conservation law, written in the form of equation (12), which gave him

$$\sum_{\nu} \frac{\partial S_{\sigma}^{\nu}}{\partial x_{\nu}} - B_{\sigma} = 0 , \tag{31}$$

where

$$S_{\sigma}^{\nu} = \sum_{\mu\tau} \left(g^{\nu\tau} \frac{\partial H\sqrt{-g}}{\partial g^{\sigma\tau}} + g_{\mu}^{\nu\tau} \frac{\partial H\sqrt{-g}}{\partial g_{\mu}^{\sigma\tau}} + \frac{1}{2}\delta_{\sigma}^{\nu} H\sqrt{-g} - \frac{1}{2} g_{\sigma}^{\mu\tau} \frac{\partial H\sqrt{-g}}{\partial g_{\nu}^{\mu\tau}} \right) . \tag{32}$$

A count of the number of equations determining the field—10 field equations, 4 adapted coordinate conditions (equation (28)) and the four equations in (32)—showed that the field was overdetermined by four conditions. This could be resolved, Einstein concluded, if the S_{σ}^{ν} identically vanished:

$$S_{\sigma}^{\nu} \equiv 0. \tag{33}$$

Since this condition did not fully determine the field equations, Einstein stipulated that H should be a homogeneous function in second order of the $g_{\sigma}^{\mu\nu}$. From this it followed unproblematically that H would have to be equal to one of, or a linear combination of, the five linearly independent terms

$$\sum g_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial x_{\sigma}} \frac{\partial g^{\sigma\tau}}{\partial x_{\tau}}, \quad \sum g^{\sigma\sigma'} g_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial x_{\sigma}} g_{\mu'\nu'} \frac{\partial g^{\mu'\nu'}}{\partial x'_{\sigma}}, \quad \sum g_{\sigma\sigma'} \frac{\partial g^{\sigma\mu}}{\partial x_{\mu}} \frac{\partial g^{\sigma'\nu}}{\partial x_{\nu}},$$

$$\sum g_{\mu\mu'} g_{\nu\nu'} g^{\sigma\sigma'} \frac{\partial g^{\mu\nu}}{\partial x_{\sigma}} \frac{\partial g^{\mu'\nu'}}{\partial x'_{\sigma}}, \text{ and } \sum g_{\alpha\beta} \frac{\partial g^{\alpha\sigma}}{\partial x_{\tau}} \frac{\partial g^{\beta\tau}}{\partial x_{\sigma}} . \tag{34}$$

Einstein then asserted that the condition (33) eventually leads to the choice of the fourth of these terms, up to a constant factor. He gave no proof, but demonstrated that this choice of H does indeed satisfy equation (33). Of course this choice of H is equivalent to the selection of H in equation (30). So Einstein's argument amounts to a new derivation of the *Entwurf* field equations.

Einstein had good reason to be pleased with this result. For it seemed to show that his theory was not just a theory of gravitation, but a generalized theory of relativity, in so far as it was concerned with establishing the widest covariance possible in its equations. His original derivation of the field equations had been based squarely on considerations in gravitation theory—that is, he sought tensor equations which would yield the correct Newtonian limit while consistent with the conservation laws. The new derivation, however, focused on covariance considerations. He had found a simple way of formulating field equations that would have exactly the maximum covariance allowed by his "hole" argument, and they led him almost directly to his original *Entwurf* field equations. As a result, he could promise to "recover the equations of the gravitational field in a purely covariant-theoretical way" and to claim to "have arrived at quite definite field equations in a purely formal way, i.e., without directly drawing on our physical knowledge of gravitation."[60] Perhaps Einstein overstated the purity of his new derivation, but it certainly is far purer than the *Entwurf* derivation.[61]

Einstein's satisfaction with his new treatment of the field equations was short-lived. He soon found that the last step in his derivation was incorrect. The condition (33) in no way required that H take on its *Entwurf* form. In effect, all that this condition required was that H be a scalar under linear coordinate transformations: it just returned an assumption that Einstein had made earlier in the derivation. We can readily confirm that this is the import of condition (33) by writing out the general expression for ΔH, which, by the usual methods, turns out to be

$$\frac{1}{2}\Delta H = \sum_{\mu\nu\sigma\alpha} \frac{1}{\sqrt{-g}} S_{\mu}^{\alpha} \frac{\partial \Delta x_{\mu}}{\partial x_{\alpha}} + g^{\nu\alpha} \frac{\partial H}{\partial g_{\sigma}^{\mu\nu}} \frac{\partial^2 \Delta x_{\mu}}{\partial x_{\sigma} \partial x_{\alpha}} . \tag{35}$$

If H is a scalar under linear coordinate transformation, then ΔH must vanish in the case in which the $\frac{\partial \Delta x_{\mu}}{\partial x_{\alpha}}$ have arbitrary non-zero values, but all the $\frac{\partial^2 \Delta x_{\mu}}{\partial x_{\sigma} \partial x_{\alpha}}$ vanish. Clearly this will only be true if condition

60. Einstein (ref. 26), 1030, 1076.

61. A new derivation of the field equations of Nordström's gravitation theory based on covariance considerations had been presented in Einstein and A. D. Fokker, "Die Nordströmsche Gravitationstheorie vom Stanpunkt des absoluten Differentialkalküls," *AP, 44* (1914), 321–328 (received 19 Feb 1914), 328. The derivation involved postulating a scalar field equation based on the Riemann curvature scalar. They concluded by speculating that a similar derivation of the "Einstein-Grossmann gravitation equations" might also be possible using the Riemann curvature tensor. They observe without further explanation that the reason given in Grossmann's section 4 of the *Entwurf* paper for the nonexistence of such a relationship between the gravitation equations and the Riemann curvature tensor does not hold up under closer consideration. All that Einstein and Grossmann had found in the *Entwurf* paper was that the then obvious methods of forging a link between the Riemann curvature tensor and the gravitation tensor seem to fail on the question of the Newtonian limit. This does not prove that such a connection does not exist and perhaps just this was the point of Einstein and Fokker's footnote. Einstein and Fokker do not seem to have doubted the correctness of the *Entwurf* field equations nor did they repudiate the general arguments against the admissibility of generally covariant field equations.

(33) is satisfied by H. Equation (35) is a generalization of equation (24). Presumably Einstein was unaware of the appearance of S_μ^α in (35) since, in his derivation of (24), there would have been no need to collect terms in $\frac{\partial \Delta x_\mu}{\partial x_\alpha}$.

All five expressions in (34) and any of their linear combinations will satisfy condition (33) since they are all scalars under linear coordinate transformation. I have been unable to find any explanation why Einstein believed that condition (33) finally led to the result that H had to take its *Entwurf* value. He asserted this result without proof in the paper and limited himself to confirming that this form of H does in fact satisfy condition (33). His correspondence and later published discussion of his work of 1914 sheds only a little light on this episode and it remains an outstanding puzzle in the history of Einstein's theory.[62]

7. THE GRADUAL DAWNING

Einstein appears to have remained satisfied with the theory he developed in 1914 through the first half of 1915. In March, April, and early May, he defended the theory wholeheartedly in an intense correspondence with Levi-Cività, who challenged Einstein's derivation of the covariance properties of his gravitation tensor. But it seems that by mid-July he was less certain. He wrote enthusiastically to Sommerfeld about his visit to Göttingen of late June and early July, where he had lectured on his theory.[63] But he was less enthusiastic about

62. Most promising of these later comments comes in a letter to Hilbert of 30 Mar 1916 (EA 13 097), in which Einstein discussed a mistake in his "work of 1914" that Hilbert had pointed out to him. Einstein noted that under the infinitesimal (coordinate?) transformation Δ the relation

$$\Delta g_\sigma^{\mu\nu} = \frac{\partial}{\partial x_\sigma}(\Delta g^{\mu\nu}) \tag{36}$$

does not hold. "Therefore," he said, "there is no variation in the sense of variational calculations that correspond to the change Δ." This cannot be the confession of a simple blunder in his calculations that might have explained the error in question. For the Einstein of 1914 knew that (36) does not hold for infinitesimal coordinate transformations; he gives the correct relation as his equation (63a) of ref. 32, and uses it correctly throughout the paper. Rather, the problem seems to be associated with the proof of the important result (29). Writing to de Sitter on 23 Jan 1917 (EA 20 540), Einstein placed a mistake, first found by Hilbert but otherwise unspecified, as somewhere in this proof. There Einstein had divided the variation of the field δ into two parts, the second of which, δ_2, was a four-parameter variation that could be generated mathematically by an infinitesimal coordinate transformation. Therefore, the variation δ_2 cannot satisfy (36), whereas the variation δ must, if the gravitation tensor of equation (23) is to be derived from it by the usual methods. Thus, in short, we can say that δ_2 is "no variation in the sense of variational calculations."

63. Einstein to Sommerfeld, 15 Jul 1915 (ref. 28), 30.

Sommerfeld's proposal that one or two papers on general relativity be included in a new edition of *Das Relativitätsprinzip,* the well-known collection of original papers in the development of relativity theory. Einstein wrote that he would prefer to see the volume left unchanged since none of the current presentations of the theory was "complete."

By mid-October Einstein's points of dissatisfaction with his theory had grown in number and intensity. They soon culminated in some of the most agitated and strenuous weeks of his life, in which generally covariant field equations were discovered, or perhaps, rediscovered. In a letter of January 1, 1916, Einstein recounted the events of these months to Lorentz:[64]

> The gradually dawning knowledge of the incorrectness of the old gravitational field equations gave me a rotten time last autumn. I had already found earlier that the perihelion motion of Mercury was too small. In addition, I found that the equations were not covariant for substitutions which corresponded to a uniform rotation of the (new) reference system. Finally I found that my approach of last year to the determination of Lagrange's function H of the gravitational field was illusory throughout, since it could be easily modified so that one needed to apply no limiting condition at all to H, so that it could have been chosen quite freely. Thus I came to the conviction that the introduction of adapted systems was a false path and a more far-reaching covariance, where possible *general* covariance, must be demanded.

Einstein gave a similar account of this dawning in an earlier letter to Sommerfeld.[65] There he noted that the old theory gave a figure of 18" of arc per century rather than 45" for the perihelion motion of Mercury.

The first result Einstein mentioned, the failure of his theory to account for the anomalous motion of Mercury's perihelion, might well have been known to him from the earliest days of the *Entwurf* theory. One of his earliest hopes for his new work on gravitation, as communicated in a letter of December 24, 1907 to Konrad Habicht, was that it might account for this anomaly.[66] The question seems to have arisen again early in 1915 in connection with the Berlin astronomer Freundlich, who had been the first to attempt astronomical tests of Einstein's new theory. In a postcard of March 1915 to Freundlich, Einstein confirmed that, according to his *Entwurf* theory, matter at rest can only yield a g_{44} field, which proved that "a g_{11} field cannot come into consideration in the problem of the planets."[67] This shows that Einstein still believed that a static field had to be spatially flat. But now he

64. EA 16 445.
65. Einstein to Sommerfeld, 28 Nov 1915 (ref. 28), 32–36.
66. EA 12 445.
67. Postmarked 19 Mar 1915, EA 11 208.

regarded it as a theorem of his *Entwurf* theory.

The second result that Einstein mentioned was his discovery that a uniform rotation of the coordinate axes did not belong to the justified coordinate transformations of his *Entwurf* field equations. We know that Einstein believed such transformations to be justified in 1914, although he never presented a proof. The Einstein Project has recently acquired a copy of a single page in Einstein's handwriting that bears on this matter.[68] Einstein here wrote out the *Entwurf* field equations. Beneath each of its terms, he put the values that their (4,4) components reduce to in a Minkowski metric viewed from uniformly rotating coordinates. In three dimensional spacetime, the transformation used seems to have been

$$t' = t,\ r' = r,\ \Theta' = \Theta - \omega t\,, \tag{37}$$

where the nonrotating coordinates are unprimed, rotating coordinates primed and the coordinates have their usual meaning.

Einstein's calculation is for the special weak-field case of small angular velocity ω and regions close to the axis of rotation. The results show that the field equations do not hold in the rotating case. Surrounding this calculation, in a way that indicates that it was added later, is the draft of a letter to Ministerial Director Naumann. This letter can be dated by content to late November or perhaps early December 1915. If the calculations are coeval with the letter, they were made around the time when Einstein returned to seek generally covariant field equations. The reverse side of the document contains some calculations on the form of the Minkowski metric in a uniformly rotating coordinate system. Einstein concentrated on the spatial part of the metric and deviated from the normal practice of equations (37) by leaving the radial coordinate in the rotating system r' an undetermined function of the original nonrotating radial coordinate r.

The relationship—if any—between these calculations and those on the first side is unclear. Perhaps Einstein was investigating some problem in rotating coordinates, for example the spatial geometry on a rotating disk, and perhaps this investigation led him to check whether the transformation to rotating coordinates was in fact justified. Or perhaps the discovery that the particular transformation (37) was not justified led him to try to find another transformation to rotating coordinates that was justified—hence the presumably unsuccessful examination of a more general transformation in which r' is an undetermined function or r. What does remain a puzzle is how Einstein could have overlooked for long the result that transformation (37) is not

68. I am grateful to John Stachel for drawing this document to my attention.

justified—if he in fact did. Or perhaps he knew that it did not hold, but was not disturbed to find that just one of many possible transformations to rotating coordinates is not justified.

The third result that Einstein described to Lorentz as precipitating his return to the search for generally covariant field equations was his discovery that his new derivation of 1914 did not actually determine H. He had found that an easy modification of his considerations no longer led to any restricting conditions on H. Lorentz would have been quite familiar with this last result, for in a letter of October 12, 1915, Einstein had described to him exactly what this modification was.[69] In his letter, Einstein recognized the condition (33) amounted only to the requirement that H be a scalar under linear coordinate transformation. He described how he missed this in 1914 since he had assumed this property for H at the beginning of the derivation.

In his modified derivation of the field equations in this letter of October 1915, he proceeded exactly as in 1914, but without this assumption. He found that the condition for adapted coordinate systems was equation (31) rather than (28). This immediately clarified the status of equation (33). Einstein continued to write the field equations that arose from the gravitation tensor in equation (23) as

$$-\sum \frac{\partial}{\partial x_\sigma}\left(g^{\nu\lambda}\frac{\partial Q}{\partial g^{\mu\nu}_\sigma}\right) = k\,\mathfrak{T}^\lambda_\mu + \left(-\sum_\nu g^{\nu\lambda}\frac{\partial Q}{\partial g^{\mu\nu}} - \sum_{\nu\sigma} g^{\nu\lambda}_\sigma \frac{\partial Q}{\partial g^{\mu\nu}_\sigma}\right), \qquad (38)$$

where $Q = H\sqrt{-g}$; and he expressed the conservation laws as

$$\sum_\lambda \frac{\partial}{\partial x_\lambda}\left(\mathfrak{T}^\lambda_\mu + \mathrm{t}^\lambda_\mu\right) = 0\,, \qquad (39)$$

where

$$\mathrm{t}^\lambda_\mu = \frac{1}{2\kappa}\sum_{\sigma\nu}\left(-g^{\nu\sigma}_\mu \frac{\partial Q}{\partial g^{\nu\sigma}_\lambda} + Q\delta^\lambda_\mu\right).\text{[70]} \qquad (40)$$

Einstein then required that the source term of the field equations, the right-hand side of (38), be equal to $\kappa(\mathfrak{T}^\lambda_\mu + \mathrm{t}^\lambda_\mu)$. It follows directly from equation (40) that this will only be true if condition (33) is satisfied. (This result is interesting in itself. In effect, Einstein showed that the field equations can always be written with a source term of this form as long as H is a scalar under linear coordinate transformations. So this result applies to his final generally covariant field equations as well.)

69. EA 16 442.

70. This form of the conservation law follows from a contraction of the field equations with $g^{\mu\nu}_\sigma$ and substitution into the conservation laws in the form of equation (12).

Einstein then completed what was to be his last derivation of the *Entwurf* field equations by noting that the choice of H as its *Entwurf* form was dictated by the requirements of the Newtonian limit. This must have been quite a setback for Einstein, for, as we have seen, he had taken great pride in he fact that his earlier derivation of 1914 of the *Entwurf* field equations seemed to be based only on covariance arguments and did not need to draw directly on any physical knowledge of gravitation.

In his letter to Lorentz, Einstein did not explain how the requirement of the Newtonian limit was to be applied. Presumably he meant that the gravitation tensor had to have the form (3). If so, then he was wrong on two counts. First, equation (3) does not quite uniquely determine the form of H as its *Entwurf* form, the fourth of those listed in (34). It admits a limited number of alternatives. For example, the gravitation tensor resulting from taking H as the fourth plus an arbitrary constant times the difference of the third and the fifth also has this property. Second, as Einstein soon discovered, Newtonian theory can still be obtained as a limiting case if we dispense with the restrictive condition (3).

But Einstein's work on this derivation had not been entirely in vain, for it had brought him both temporally and conceptually closer than ever before to a generally covariant theory. If we ignore Einstein's last fatal step, we find that the mathematical apparatus set up here by Einstein and, earlier, by Grossmann, can be used almost unchanged in the final generally covariant theory. For if we make the now familiar selection

$$H = g^{\mu\nu}\left(\left\{ {\sigma \atop \mu\nu} \right\}\left\{ {\rho \atop \sigma\rho} \right\} - \left\{ {\rho \atop \mu\sigma} \right\}\left\{ {\sigma \atop \nu\rho} \right\}\right), \tag{41}$$

then the field equations of the final theory follow.[71] The adapted coordinate condition still holds, but in a degenerate form, for now all coordinate systems are adapted. In fact the adapted coordinate condition, written as either equation (28) or (31) is none other than the contracted Bianchi identities.

Einstein clearly came to recognize these results, for they comprise the major part of his paper of late 1916 on a Hamiltonian formulation of the general theory of relativity.[72] By adopting a gravitational field

71. $\left\{ {\sigma \atop \mu\nu} \right\}$ is the Christoffel symbol of the second kind and summation over repeated indices is implied. This expression results from the Riemann curvature scalar after terms in the second derivatives of the metric tensor are separated out as a total divergence term. See, for example, P. Dirac, *General theory of relativity* (New York, 1975), 48.

72. Einstein, "Hamiltonsches Prinzip und allgemeine Relativitätstheorie," AW, *Sb,* 1916, 1111–1116 (received 26 Oct 1916).

action density based on the Riemann curvature scalar, he arrived at an H of the form of (41). Since this H is a scalar under linear coordinate transformations, he arrived at condition (33) and then directly at condition (28). With this choice of H, this last condition is equivalent to the contracted Bianchi identities, although Einstein was unaware presumably of the connection to the uncontracted Bianchi identities at this time.[73] He then proceeded to the field equations and conservation laws written in a form similar to equations (38), (39), and (40), but now using condition (28) to derive the conservation laws from the field equations and their covariance properties.

It is hard to imagine that Einstein was unprepared for the ease with which his formalism of 1914 could be applied to his final generally covariant theory. In 1914, in the paper in which he had first introduced the adapted coordinate condition, he remarked—prophetically—that, were this condition to be generally covariant, then all coordinate systems would be adapted and that this consequence would not compromise any step of his proof.[74] But, he continued, these conditions were not generally covariant in the *Entwurf* theory; for if they were, the fully contracted gravitation tensor would have to be none other than the Riemann curvature scalar and it was not.

Hilbert, through his important paper of November 1915, is generally thought of as introducing the comprehensive use of these action principles to the theory.[75] My analysis shows that although Einstein might have drawn some of his work of 1916 in this area from Hilbert's, his basic mathematical apparatus and even the notation itself had its ancestry in his own work earlier in 1914 and 1915.

8. "THE FINAL EMERGENCE INTO THE LIGHT"[76]

By mid-October 1915, Einstein was no longer satisfied with his theory. Presumably he had been disappointed that it had not accounted for the anomalous motion of Mercury. Perhaps this shortcoming had become all the more acute with the difficulties Freundlich faced in his attempts to set up and carry out astronomical tests of the theory. Then Einstein convinced himself that transformations to rotating coordinate systems were not "justified," which must have compromised his belief

73. Mehra (ref. 18), 49–50, 78, and Pais (ref. 18), 274–278, discuss the delay in recognition of this connection.

74. Einstein and Grossmann (ref. 52), 224–225.

75. D. Hilbert, "Die Grundlagen der Physik," Akademie der Wissenschaften, Göttingen, *Nachrichten*, 1915, 395–407.

76. Title from Einstein, "Notes on the origin of the general theory of relativity," in Einstein, *Ideas and opinions* (London, 1973), 289–290.

that his new theory extended the relativity of motion to accelerated motion. Finally, he found that all his elegant manipulation of covariance requirements and adapted coordinates did not even lead him to a definite set of field equations. With doubts accumulating about the empirical, physical, and formal foundations of his theory, Einstein took drastic action. He wrote to Sommerfeld that, at this point, he gave up the notion of requiring covariance with respect to adapted coordinate systems:[77]

> After all trust in the results and methods of the earlier theory had thus given way, I saw clearly that only in a link to the general theory of covariants, i.e., to Riemann's covariant, could a satisfactory solution be found. Unfortunately I have immortalized the last errors in this struggle in the Academy papers, which I can send you soon.

The first of these "last errors in this struggle" was presented to the Prussian Academy on November 4, 1915. He divided the Ricci tensor G_{im} into the sum of two parts:[78]

$$G_{im} = R_{im} + S_{im} \ , \tag{42}$$

$$R_{im} = -\sum_{l} \frac{\partial \left\{ {im \atop l} \right\}}{\partial x_l} + \sum_{\rho l} \left\{ {il \atop \rho} \right\} \left\{ {\rho m \atop l} \right\} , \tag{43}$$

$$S_{im} = \sum_{l} \frac{\partial \left\{ {il \atop l} \right\}}{\partial x_m} - \sum_{\rho l} \left\{ {im \atop \rho} \right\} \left\{ {\rho l \atop l} \right\} . \tag{44}$$

Einstein had shown that if we restrict ourselves to coordinate transformations of determinant one, then $\sqrt{-g}$ is a scalar. From this it followed easily that S_{im} is a tensor under all such transformations and, since G_{im} is a generally covariant tensor, then R_{im} must also be a tensor under the restricted set of coordinate transformations. Einstein selected this as his gravitation tensor, and his field equations became

$$R_{\mu\nu} = -\kappa T_{\mu\nu} \ . \tag{45}$$

Why Einstein should choose this as his gravitation tensor rather than a generally covariant tensor, such as the Ricci tensor or even the Einstein tensor itself, has hitherto been a puzzle. It can now be solved by reference to my discussion of Einstein's original objections to the

77. Letter of 28 Nov 1915 (ref. 28), 32–36.

78. Einstein (ref. 23). As before, I follow Einstein in using G_{im} to refer to the Ricci tensor rather than the modern usage, in which G_{im} would refer to the Einstein tensor. For consistency, I have made Einstein's implicit summation explicit.

Ricci and related tensors as gravitation tensors. The results that led to his disillusionment with the *Entwurf* theory had left these original objections substantially intact. Einstein still expected his field equations to reduce to the form of (11) in the weak-field case. Einstein knew that the Ricci tensor reduced to the appropriate form with the application of the harmonic coordinate condition. But, as we have seen, this coordinate condition was unacceptable to him for it was inconsistent with the form of weak, static fields entailed by equation (11).

But there was a second possibility, the tensor $\mathfrak{Z}^{x}_{il}$, which is the same as R_{im} above. This reduces to the required form with the coordinate condition (10). I argued that Einstein rejected this second tensor because he found that this coordinate condition was not satisfied in a Minkowski spacetime viewed from rotating coordinates, a requirement that would ensure that the field equations retain the weak-field form of equation (11) in such rotating coordinates. But, as we have seen, by October 1915 Einstein had found that his *Entwurf* field equations, which had the required weak-field form, were not satisfied in such rotating coordinates. I conjecture that this discovery led him to reconsider the rather restrictive requirement that the field equations still have this weak-field form in such rotating systems, for he no longer had any objections to the tensor R_{im} as a gravitation tensor in his paper of November 4.

In the concluding section of this paper, Einstein drew all these elements together. He stated that his new field equations reduce to the form of equation (11) in the weak-field case with the application of coordinate condition (10). This confirms my assertion. that he still believed that his field equations must have this weak-field form and also that his choice of R_{im} as the gravitation tensor was based on the fact that they reduce to this form with the help of coordinate condition (10). Perhaps the juxtaposition is accidental, but Einstein completed the paper by noting that his new field equations are indeed covariant under transformation to rotating coordinate systems and to those in rectilinear acceleration, as required by the relativity of motion.

My account of Einstein's paper of November 4 has left a problem. The recovery of the weak-field equations described involved the application of the four coordinate conditions (10) and also the condition that coordinate transformations of determinant one only be admitted. This last condition amounts to one more coordinate condition than the four normally permitted. That Einstein did not in fact overdetermine his equations follows from his treatment of the single coordinate condition that arises from this last condition. First, he fully contracted the field equation (45) to yield

$$\sum_{\alpha\beta}\frac{\partial^2 g^{\alpha\beta}}{\partial x_\alpha \partial x_\beta}-\sum_{\sigma\tau\alpha\beta} g^{\sigma\tau}\Gamma^{\alpha}_{\sigma\beta}\Gamma^{\beta}_{\tau\alpha}+\sum_{\alpha\beta}\frac{\partial}{\partial x_\alpha}\left[g^{\alpha\beta}\frac{\partial\, lg\sqrt{-g}}{\partial x_\beta}\right]=-\kappa\sum_{\sigma}T^{\sigma}_{\sigma}\ . \qquad (46)$$

Then, using familiar variational methods to define the stress-energy tensor of the gravitational field t_μ^λ, he wrote the field equations in the mixed form

$$\sum_{\alpha\nu} \frac{\partial}{\partial x_\alpha}\left(g^{\nu\lambda}\Gamma^\alpha_{\mu\nu}\right) - \frac{1}{2}\delta^\lambda_\mu \sum_{\mu\nu\alpha\beta} g^{\mu\nu}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} = -k\left(T^\lambda_\mu + t^\lambda_\mu\right).^{79} \tag{47}$$

The coordinate divergence of the right-hand side of this equation vanishes as a result of the conservation laws. This yields the four conditions

$$\frac{\partial}{\partial x_\mu}\left[\sum_{\alpha\beta}\frac{\partial^2 g^{\alpha\beta}}{\partial x_\alpha \partial x_\beta} - \sum_{\sigma\tau\alpha\beta} g^{\sigma\tau}\Gamma^\alpha_{\sigma\beta}\Gamma^\beta_{\tau\alpha}\right] = 0\,, \tag{48}$$

which correspond to the "adapted" coordinate conditions of the former theory. Equation (48) can be solved directly to yield a single condition—that the term in square brackets be a constant—which amounts to the coordinate condition imposed by the limitation of coordinate transformations to those with a determinant of one. Einstein set the value of this constant as zero, as one is free to do with such conditions, and thus arrived at

$$\sum_{\alpha\beta}\frac{\partial^2 g^{\alpha\beta}}{\partial x_\alpha \partial x_\beta} - \sum_{\sigma\tau\alpha\beta} g^{\sigma\tau}\Gamma^\alpha_{\sigma\beta}\Gamma^\beta_{\tau\alpha} = 0\,. \tag{49}$$

These considerations resolve the problem of the overdetermination of the weak-field equations, for we can see immediately that in the weak-field case, in which terms quadratic in the derivatives of the metric tensor can be ignored, the coordinate conditions (10) entail the coordinate condition (49). Thus Einstein could introduce the conditions (10), in the closing section of his paper, as a strengthening of the condition (49).[80]

79. Einstein (ref. 23). Specifically,

$$\kappa t^\lambda_\sigma = \frac{1}{2}\delta^\lambda_\sigma \sum_{\mu\nu\alpha\beta} g^{\mu\nu}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \sum_{\mu\nu\alpha} g^{\mu\nu}\Gamma^\alpha_{\mu\sigma}\Gamma^\lambda_{\nu\alpha}\,,$$

so that

$$\kappa t = \sum_\sigma \kappa t^\sigma_\sigma = \sum_\sigma g^{\mu\nu}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}\,.$$

Note $\Gamma^\tau_{\nu\sigma} = -\left\{ {\nu\sigma \atop \tau} \right\}$.

80. Of course this weak-field assumption is itself a coordinate condition of a kind. Strictly speaking, Einstein would have to show that his five coordinate conditions were also consistent with this new constraint. This should not have been a problem, for a coordinate system in which the weak-field assumption holds is still determined only up to a four parameter infinitesimal coordinate transformation.

The conditions which Einstein derived here led to one further result that was to be of great importance. The two scalar conditions (46) and (49) could only both be true if the condition

$$\sum_{\alpha\beta} \frac{\partial}{\partial x_\alpha} \left(g^{\alpha\beta} \frac{\partial \, lg\sqrt{-g}}{\partial x_\beta} \right) = -\kappa \sum_{\sigma} T_\sigma^\sigma \qquad (50)$$

held. Einstein noted that equation (50) meant that $\sqrt{-g}$ could not be set equal to unity, for the trace of the stress-energy tensor $T = \sum_\sigma T_\sigma^\sigma$ cannot be made zero.[81]

Einstein made one other point of special interest in his communication of November 4. He described "a fatal prejudice" in his earlier work: he had been induced to take the quantity

$$\frac{1}{2} \sum_{\mu} g^{\tau\mu} \frac{\partial g_{\mu\nu}}{\partial x_\sigma}$$

for the components of the gravitational field $\Gamma_{\nu\sigma}^\tau$.[82] He now recognized that the Christoffel symbol of the second kind, or, to be exact, its negation, $-\left\{ {\nu\sigma \atop \tau} \right\}$, was the quantity he should have selected and proceeded to argue for this new choice. Einstein did not explain why this prejudice was so fatal. A comparison of the equations of his *Entwurf* theory and the new theory may make it clearer. A weak correspondence between the forms of the equations of each theory appears if they are written in terms of $\Gamma_{\nu\sigma}^\tau$, where these components have the appropriate forms as specified above. The action densities and the gravitational field stress-energy tensors of both theories take on exactly the same form. Moreover, the second derivative terms of each of the two theories' gravitation tensors take on the same form, $\frac{\partial \Gamma_{\nu\sigma}^\tau}{\partial x_\tau}$. In the *Entwurf* case, this term is simply the d'Alembertian of the metric tensor. In the case of the new theory, however, this expression contains other second derivatives of the metric tensor, which Einstein had tried so hard to eliminate

81. Presumably Einstein referred to the general case in which the source of the field is unspecified. Then, from equation (50), $\sqrt{-g}$ cannot have any constant value whose coordinate derivatives all vanish. Einstein's procedure is not "incoherent," contrary to the assertion of J. Earman and C. Glymour, "Einstein and Hilbert: Two months in the history of general relativity," *Archive for history of exact sciences, 19* (1978), 291–308, on 298–299. For further evidence, note that Einstein chose not to use the familiar covariant divergence of the stress-energy tensor in his conservation laws, but to replace it by a different but closely related quantity, which he demonstrated to be covariant under coordinate transformations with a determinant of one. The distinction between the two divergences drops away when $\sqrt{-g} = 1$, so Einstein's nonstandard choice in ref. 28 did not become important in his work over the following weeks.

82. Einstein (ref. 28), 782–783.

some three years earlier when trying to recover a Newtonian limit. So Einstein could write to Sommerfeld on November 28, 1915, that this final approach made his final field equations "the simplest conceivable since one is not tempted to transform them by multiplying out the symbols with the intention of more general interpretation."[83]

(It does not seem that pursuing this line of thought will help us further delineate the path Einstein took in 1912 and 1913. For Einstein's comments have all the flavor of an after-the-fact rationalization. Note in particular that Einstein did not introduce the notion of the "components of the gravitational field" into his *Entwurf* theory until 1914.[84])

Einstein did not remain satisfied with his theory of November 4 for very long. During the following week he found a simple modification to his theory that left its mathematical machinery essentially untouched but now brought field equations that were at last generally covariant. It seems likely that the modification occurred to him as a result of a reexamination of equation (50). From his standpoint on November 4, it followed that $\sqrt{-g}$ could not in general be a constant. Certainly there seemed to be no physical reason in the theory for such a limitation.

However, the equation can be read in a second way. We can regard it not as placing a limit on the form of $\sqrt{-g}$, but as restricting the value of T. Specifically, if $\sqrt{-g}$ has a constant value, then T must vanish. Now this latter restriction does admit a simple physical interpretation, which Einstein was to seize with enthusiasm. If all matter were electromagnetic in nature, then this condition would be automatically satisfied. As is well known, the trace of the stress-energy tensor of an electromagnetic field is always zero.

On November 11 Einstein introduced his latest formulation of the theory with this hypothesis about the electromagnetic nature of all matter.[85] He asserted that the hypothesis made possible the final step to generally covariant field equations. For these equations he wrote

$$G_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{51}$$

From them he could recover the equations of November 4 and still use all their associated mathematical machinery by applying the coordinate

83. Ref. 65. Einstein also mentioned his "prejudice" in a letter to Lorentz, 1 Jan 1916, EA 16 445.

84. Einstein (ref. 32), 1058.

85. Einstein, "Zur allgemeinen Relativitätstheorie (Nachtrag)," AW, *Sb*, 1915, 799–801 (read 11 Nov 1915). That this hypothesis resulted from a reexamination of the equations in the earlier theory, rather than from some external source, is suggested by a footnote (ibid., 800): "At the writing of the earlier communication, the admissibility in principle of the hypothesis $\sum T_\mu^\mu = 0$ had not yet come to consciousness."

condition

$$\sqrt{-g} = 1, \tag{52}$$

which in turn entailed the condition that only coordinate transformations of determinant one be admitted. Under condition (52), S_{im} vanished and his new field equation reduced to that of November 4 in form. He concluded his note of November 11 by noting that equation (50) entailed the vanishing of T if the condition (52) held.

In the unkind gaze of historical hindsight, Einstein's "mistake" seems simple. He had been forced to admit a dangerous conjecture about the nature of matter in order to conceal the fact that he had missed the now familiar trace term in the "correct" field equations. They are

$$G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} G = -\kappa T_{\mu\nu} , \tag{53}$$

where G is the trace of $G_{\mu\nu}$, or, in an equivalent form,

$$G_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) . \tag{54}$$

These field equations are consistent with the conservation laws without further hypothesis since the covariant divergence of the left-hand side of equation (53) vanishes identically. Transferring this property to the right-hand side gives the conservation laws the vanishing of the covariant divergence of the stress-energy tensor. Now Einstein's field equations of November 11 did not have this property. But the assumption that $T = 0$ converts equations (53) and (54) into those field equations and brings them into accord with the conservation laws. That the assumption does this is not surprising since it came originally from equation (50), which in turn was derived with the help of the conservation laws.

This simple analysis completely misses what Einstein had achieved with his modification of November 11. He had finally succeeded in finding generally covariant field equations that reduced to the weak-field form of equation (11). This reduction could be effected by the application of coordinate conditions (10) and the new condition (52). That these five conditions did not overdetermine the field followed almost immediately from his paper of November 4. There, as we have seen, Einstein showed the consistency in the weak-field case of using the four conditions (10) and also limiting coordinate transformations to those with a determinant of one. This latter limitation, which amounted to the limitation to coordinate systems in which $\sqrt{-g}$ behaves like a scalar, was strengthened in the following note to the requirement that $\sqrt{-g}$ be a constant, specifically unity.

An examination of the relevant equations of Einstein's paper of November 4 (equations (46) to (50) here) shows that use of condition (10) strengthened with (52) does not compromise his consistency arguments, provided that we are willing to accept the new constraint on the field source term of $T = 0$. The adoption of the extra trace terms of equations (53) or (54) would have been out of the question, for they would have destroyed the hard-won and finely-tuned agreement between the field equations and their weak-field limit. We can now appreciate why field equations (51) seemed the only possible generally covariant field equations to Einstein and thus why the adoption of the hypothesis $T = 0$ seemed a small price to pay for the final achievement of general covariance. Indeed, in the note of November 11, he seemed pleased to regard the information it contained as an unexpected bonus from the requirement of general covariance.

What still separated Einstein from his final field equations was not a simple oversight, but the same almost untouched misconceptions about the weak-field limit that he had had three years earlier. I argued that he expected the field equations to reduce to equation (11) in the weak-field case because of their formal simplicity and because they in turn enabled a simple reduction of the ten gravitational potentials of the full theory to a single Newtonian potential, in a simple static-field case. The naturalness of such a reduction was corroborated by Einstein's belief that, on the basis of quite separate arguments, the number of gravitational potentials underwent a similar reduction in the case of a general static field.

The final realization that his ideas on the behavior of weak and static fields were excessively restrictive and not justified by experience came over the two weeks following November 11. What seems to have catalyzed this realization was his calculation of the orbit of Mercury. He turned to this task immediately after he had arrived at the modified field equations and was able to present his results to the Prussian Academy just one week later, on November 18.[86] In this communication, Einstein used a method of successive approximations to solve his field equations for the gravitational field of the sun, that is for the weak, static, spherically symmetric, source-free case, with Minkowskian values at spatial infinity. He presented a solution for this case that satisfied the condition $\sqrt{-g} = 1$, so that the field equations he was solving, those of November 11, reduced in form to those of November 4. Einstein had already shown that these latter field equations yield the

86. Einstein, "Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie," AW, *Sb.* 1915, 831–839 (read 18 Nov 1915). Even Hilbert was impressed at the speed with which Einstein calculated the perihelion motion, as he told Einstein in his congratulatory postcard of 19 Nov 1915, EA 13 054.

weak-field equations (11). Provided that the central mass is small, these equations solve to yield a spatially flat field. However, such a weak-field solution was no longer allowed to Einstein, for his solution of the field equations had to satisfy the condition $\sqrt{-g} = 1$. Thus Einstein was forced to a solution that had nonconstant $g_{11}, g_{12}, \cdots, g_{33}$ even in the first approximation. He commented on this crucial new development:[87]

> From our theory it follows that, in the case of masses at rest, the components g_{11} to g_{33} are different from zero already in quantities of the first order. We shall see later that through this no contradiction arises with Newton's law (in the first approximation).

This demonstration followed soon. Einstein was able to show that in quantities of the first order of smallness the equations of motion of a slow moving mass point reduce to those of Newtonian gravitation theory. Although it was not stated in this communication, this demonstration rested on the fact that only the g_{44} component of the metric tensor was used in the construction of these equations in the first approximation. I have quoted Einstein's communication of this "most remarkable" result to Besso in December 1915. Einstein seems to have remained impressed by it, for he still described this feature of the equations of motion as "remarkable" in his summary article on the theory written early the next year.[88] Again, in his lectures of 1921 published as *The meaning of relativity,* he attributed the absence of an earlier recognition of the tensorial nature of the gravitational potential to this same feature of the equations of motion.[89] This comment can be applied to Einstein's own early treatment of weak and static fields.

Einstein's calculation of the orbit of Mercury was of crucial significance in the historical development of the theory. It gave the theory its first convincing empirical success. It was instrumental in freeing Einstein from his long-standing misconceptions about static and weak fields. It forced him to deal with a weak, static field, whose g_{11} to g_{33} components were not constant. In the communication of November 18, Einstein had already begun to tease out the implications of this revelation. He noted that his new theory predicted twice the deflection of a ray of starlight grazing the sun than that predicted by his earlier theory—including the field equations of November 4.[90]

Most significantly, Einstein was no longer constrained by the requirement that his field equations reduce to the weak-field equation

87. Einstein (ref. 86), 834.
88. Einstein (ref. 51), 817; Einstein to Besso (ref. 34).
89. Einstein (ref. 30), 86.
90. Einstein (ref. 86), 834.

(11). It was now no longer necessary for the weak-field equations to give in the appropriate static cases a metric of the form (1); field equations with additional terms to those in equation (11) could be contemplated. Einstein no longer had to adopt his field equations of November 11 and the associated $T = 0$ condition as the only possible generally covariant field equations. At last he was free to entertain field equations of the form of equations (53) and (54). Einstein may have realized this possibility very soon after completion of his paper on Mercury's motion. Although this communication still used the field equations of November 11 and the hypothesis $T = 0$, a footnote on its first page promised a new communication in which the hypothesis would be shown to be superfluous. The import of this footnote is not entirely clear since it makes no reference to new field equations. In any case, Einstein's next communication, presented to the Prussian Academy on November 25, gave as the results of nearly three years of labor, Einstein's final field equations (54).[91] He could also note—presumably with some relief—that this final modification did not affect the source-free form of the field equations and the resulting explanation of the anomalous motion of Mercury.

The question of the exact path that Einstein followed from November 11 to November 25 has become of some interest to historians of relativity. In his communication on the later date, Einstein dealt with what he called "the reasons that gave rise to my introduction of the second term on the right-hand side of the field equations [54]."[92] These, he tells us, arose in considerations analogous to those dealt with in equations (46) to (50). He noted that his new field equations, when fully contracted, become

$$\sum_{\alpha\beta} \frac{\partial^2 g^{\alpha\beta}}{\partial x_\alpha \partial x_\beta} - \kappa(T+t) = 0 , \tag{55}$$

which corresponds to the earlier equation (46). However, he observed, in this new equation, both T_σ^λ and t_σ^λ appear in a fully symmetrical way, unlike the case of equation (46). Further reduction of the field equations with the conservation laws in the manner that produced equations (48) now yields

$$\frac{\partial}{\partial x_\mu}\left(\sum_{\alpha\beta} \frac{\partial^2 g^{\alpha\beta}}{\partial x_\alpha \partial x_\beta} - \kappa(T+t)\right) = 0 . \tag{56}$$

In the context of the theory of November 11, Einstein found that the

91. Einstein, "Die Feldgleichungen der Gravitation," AW, *Sb*, 1915, 844–847 (communicated 25 Nov 1915).

92. Ibid., 846.

introduction of the hypothesis $T = 0$ was needed to bring the equations (46) and (48) into accord. He noted two weeks later that this hypothesis was no longer necessary. The corresponding equation (55) actually entails the corresponding equation (56).

This necessity indicates that Einstein added the new trace term of his field equations of November 25 in an explicit attempt to modify conditions (46) and (48) so that the hypothesis $T = 0$ would no longer be necessary. Further, his observation on equation (55) suggests a natural way in which Einstein could have found exactly what this modification to the field equations should be: they should be modified so that both T_σ^λ and t_σ^λ appear in a symmetrical way. That this is not the case with the field equations of November 11 becomes especially clear if we write them in mixed form in a coordinate system in which $\sqrt{-g} = 1$, in which case they take the form of equation (47). The second term on the right-hand side is equal to

$$-\frac{1}{2}\delta_\mu^\lambda \kappa t ,$$

and it is in this term that the asymmetry lies. If this term is replaced by

$$-\frac{1}{2}\delta_\mu^\lambda \kappa (T + t) ,$$

then the field equations become fully symmetrical in T_σ^λ and t_σ^λ and equivalent to the final equations of November 25.

The preceding considerations suggest a natural path for Einstein to have followed between November 4 and the condition (46) and (48) and November 25 and the final field equations. For the earlier field equations written exactly in the form of equation (47) appear in Einstein's paper of November 4 as a part of his derivation of conditions (46) and (48). Moreover, we know that Einstein used arguments of exactly this type at that time. In his review article of the theory written early the next year, he generated his field equations by first writing them in their source-free form,

$$\frac{\partial}{\partial x_\alpha}\left(g^{\sigma\beta}\Gamma_{\mu\beta}^{\alpha}\right) = -\kappa\left(t_\mu^\sigma - \frac{1}{2}\delta_\mu^\sigma t\right) , \tag{57}$$

and then generalizing to their complete form,

$$\frac{\partial}{\partial x_\alpha}\left(g^{\sigma\beta}\Gamma_{\mu\beta}^{\alpha}\right) = -\kappa\left((t_\mu^\sigma + T_\mu^\sigma) - \frac{1}{2}\delta_\mu^\sigma (t + T)\right) , \tag{58}$$

by requiring that $t_\mu^\sigma + T_\mu^\sigma$ replace t_μ^σ everywhere.[93] Of course the field

93. Einstein (ref. 51), 806–807.

equations in the form (58) are identical with those to be produced by the modification of equation (47) described above.

It is now known that Hilbert in Göttingen was able to arrive at substantially the same gravitational field equations as Einstein and that these equations were presented to the Göttingen Academy on November 20, 1915, five days prior to Einstein's presentation of his final field equations to the Berlin Academy.[94] Building on the results of Einstein and Mie, Hilbert used now familiar variational techniques to derive both gravitational and electromagnetic field equations and the associated conservation laws from a combined action density, whose gravitational part was the Riemann curvature scalar. His gravitational field equations took the form (53), with the added constraint that the stress-energy tensor on the right-hand side was purely electromagnetic in nature and written in terms of a derivative of the electromagnetic action density with respect to the metric tensor. Apart from the inevitable and fruitless question of priority of discovery, there has been some speculation that Einstein's final formulation of his field equations may have been influenced by a knowledge of Hilbert's field equations. In November 1915, Einstein virtually suspended his usual correspondence, but maintained an active exchange with Hilbert, in which it is possible that Hilbert communicated his field equations to Einstein sometime between November 15 and 18.[95]

We do not know the exact extent of Einstein's knowledge of Hilbert's work in November 1915. Whatever it may have been, however, it seems unlikely that it contributed in any decisive way to Einstein's final formulation of his field equations. I have tried to show here how Einstein's final steps were self-contained. His omission of the familiar trace term in his field equations of November 11 was not the consequence of a simple oversight that could be remedied by a glance at Hilbert's equations. Einstein had good reasons for not admitting any such additional terms. When he realized that these reasons were incorrect, he introduced the new terms by a path that can be fairly readily reconstructed.

There was a brief period of coolness between Einstein and Hilbert immediately after their November correspondence. Einstein's former assistant, E. G. Strauss, attributes this coolness to Einstein's feeling that Hilbert had perhaps unwittingly plagiarized some of Einstein's earlier ideas on the theory from lectures he gave in Göttingen in 1915.[96] A

94. Hilbert (ref. 91); see also Mehra (ref. 18).

95. See especially Earman and Glymour (ref. 81). They and Pais (ref. 21), 257–261, also outline the extant contents of this exchange, although Pais acquits Einstein of the charge of plagiarism.

96. Pais (ref. 18), 261.

recently discovered letter of Einstein's to his good friend Zangger in late November or early December 1915 supports Strauss' view, although the letter does not mention Hilbert by name.[97] Further, Einstein seems to have been unfamiliar with the detailed content of Hilbert's communication of November 20 as late as May 1916, even though he had stayed with the Hilberts some months earlier. Einstein reopened his correspondence with Hilbert that May with a plea for help in understanding Hilbert's paper, which he had to review in a coming colloquium in Berlin.[98] It was only at the end of this exchange that Einstein felt he could state with certainty that his and Hilbert's results agreed.

Einstein's ignorance of the details of Hilbert's work before May 1916 extended to Hilbert's result that was of crucial significance in the context of Einstein's final field equations. Hilbert wrote his gravitational field equations in terms of the variational derivative of the Riemann curvature scalar density with respect to the components of the metric tensor.[99] In the line immediately following, he stated without detailed proof that this variational derivative is equal to what we now know as the tensor density corresponding to the Einstein tensor, the tensor on the left-hand side of equation (53). Einstein could not have been aware of this result the following January. Then he wrote to Lorentz that the theory would gain greatly in clarity if a Hamiltonian formulation could be found for its field equations in their general form, that is, in their form prior to the imposition of the constraint $\sqrt{-g} = 1$.[100] This, of course, was what Hilbert had already done. Einstein even described to Lorentz how it appeared to him that the appropriate action density should be the Riemann curvature scalar density, the result Hilbert had already stated, and began to map out the derivation of the field equations from it.[101]

97. H. A. Medicus, "A comment on the relations between Einstein and Hilbert," *American journal of physics*, forthcoming.

98. Einstein to Hilbert, May 25 and 30 and Jun 2, 1916, EA 13 099, 102, 104; Hilbert to Einstein, 27 May 1916, EA 13 056. Einstein complained about the obscurity of Hilbert's work to Hilbert (EA 13 102) and, in stronger terms, to Ehrenfest, where accused Hilbert of having "pretensions of being a superman by hiding [his] methods" (24 May 1916, EA 9 378).

99. Hilbert (ref. 75), 404.

100. A. Einstein to H. A. Lorentz, 17 and 19 Jan 1916, EA 16 447, 449. Einstein's letter to Sommerfeld of 9 Dec 1915 again suggests a limited knowledge of Hilbert's work (ref. 28), 37.

101. Lorentz, "On Einstein's theory of gravitation," Academy of Sciences, Amsterdam, *Proceedings*, *19* (1916), 1341–1369, *20* (1916), 2–34 (communication of Feb 26 and Apr); Lorentz described his success in establishing this result in Lorentz to Einstein, 6 Jun 1916, EA 16 451. Einstein's uncertainty about this result would explain why it did not appear in the review article (ref. 61), where he used methods very similar to those of November 1915 to derive his field equations and to establish their consistency with the

Einstein wrote Lorentz ~~Sommerfeld~~ early in 1915:[102]

> There are two ways that a theoretician goes astray
> 1) The devil leads him around by the nose with
> a false hypothesis
> (For this he deserves pity)
> 2) His arguments are erroneous and ridiculous
> (For this he deserves a beating).

I have tried to show that Einstein went astray in the first way, rather than in the second. The cause of his straying is inseparable from his characteristic methods. On the one hand stood his relentless and uncompromising insistence on certain fundamental physical principles—the requirement of the Newtonian limit, the conservation laws, physical causality. On the other hand was the remarkable flexibility that enabled Einstein to reject even the most cherished of notions if his basic principles seemed to call for it—in this case he was prepared to forfeit general covariance. A lesser physicist might have compromised and faltered. But, eventually and perhaps inevitably, Einstein's same uncompromising relentlessness enabled him to weed out the false hypotheses that had misled him and brought him to his goal, his general theory of relativity.

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conservation laws. These same results can be established much more easily from a variation principle that uses the Riemann curvature scalar as the gravitational field action density and, as Einstein was to show in ref. 72, this deduction could be achieved with little effort by making use of the mathematical machinery of his investigations of 1914. Presumably Einstein also preferred this latter method: EA 2 077, an early version of ref. 88, seems to have been intended first to replace the derivation in ref. 61 and then to be an appendix to that paper.

102. A. Einstein to H. A. Lorentz, 2 Feb 1915. A copy of this letter has been recently acquired by the Einstein Project.