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EINSTEIN, NORDSTRÖM AND THE EARLY DEMISE OF SCALAR, LORENTZ COVARIANT THEORIES OF GRAVITATION

1. INTRODUCTION

The advent of the special theory of relativity in 1905 brought many problems for the physics community. One, it seemed, would not be a great source of trouble. It was the problem of reconciling Newtonian gravitation theory with the new theory of space and time. Indeed it seemed that Newtonian theory could be rendered compatible with special relativity by any number of small modifications, each of which would be unlikely to lead to any significant deviations from the empirically testable consequences of Newtonian theory.[1] Einstein's response to this problem is now legend. He decided almost immediately to abandon the search for a Lorentz covariant gravitation theory, for he had failed to construct such a theory that was compatible with the equality of inertial and gravitational mass. Positing what he later called the principle of equivalence, he decided that gravitation theory held the key to repairing what he perceived as the defect of the special theory of relativity—its relativity principle failed to apply to accelerated motion. He advanced a novel gravitation theory in which the gravitational potential was the now variable speed of light and in which special relativity held only as a limiting case.

It is almost impossible for modern readers to view this story with their vision unclouded by the knowledge that Einstein's fantastic 1907 speculations would lead to his greatest scientific success, the general theory of relativity. Yet, as we shall see, in

1 In the historical period under consideration, there was no single label for a gravitation theory compatible with special relativity. The Einstein of 1907 would have talked of the compatibility of gravitation and the *principle* of relativity, since he then tended to use the term "principle of relativity" where we would now use "theory of relativity". See (CPAE 2, 254). Minkowski (1908, 90) however, talked of reform "in accordance with the world postulate." Nordström (1912, 1126), like Einstein, spoke of "adapting ... the theory of gravitation to the principle of relativity" or (Nordström 1913, 872) of "treating gravitational phenomena from the standpoint of the theory of relativity," emphasizing in both cases that he planned to do so retaining the constancy of the speed of light in order to distinguish his work from Einstein's and Abraham's. For clarity I shall describe gravitation theories compatible with special relativity by the old-fashioned but still anachronistic label "Lorentz covariant." It describes exactly the goal of research, a gravitation theory whose equations are covariant under Lorentz transformation. For a simplified presentation of the material in this chapter, see also (Norton 1993).

Jürgen Renn (ed.), *The Genesis of General Relativity*
Vol. 3: *Theories of Gravitation in the Twilight of Classical Physics*. *Part I*.

1907 Einstein had only the slenderest of grounds for judging all Lorentz covariant gravitation theories unacceptable. His 1907 judgement was clearly overly hasty. It was found quite soon that one could construct Lorentz covariant gravitation theories satisfying the equality of inertial and gravitational mass without great difficulty. Nonetheless we now do believe that Einstein was right in so far as a thorough pursuit of Lorentz covariant gravitation theories does lead us inexorably to abandon special relativity. In the picturesque wording of Misner et al. (1973, Ch.7) “gravity bursts out of special relativity.”

These facts raise some interesting questions. As Einstein sped towards his general theory of relativity in the period 1907–1915 did he reassess his original, hasty 1907 judgement of the inadequacy of Lorentz covariant gravitation theories? In particular, what of the most naturally suggested Lorentz covariant gravitation theory, one in which the gravitational field was represented by a scalar field and the differential operators of the Newtonian theory were replaced by their Lorentz covariant counterparts? Where does this theory lead? Did the Einstein of the early 1910s have good reason to expect that developing this theory would lead outside special relativity?

This paper provides the answers to these questions. They arise in circumstances surrounding a gravitation theory, developed in 1912–1914, by the Finnish physicist Gunnar Nordström. It was one of a number of more conservative gravitation theories advanced during this period. Nordström advanced this most conservative scalar, Lorentz covariant gravitation theory and developed it so that it incorporated the equality of inertial and gravitation mass. It turned out that even in this most conservative approach, odd things happened to space and time. In particular, the lengths of rods and the rates of clocks turn out to be affected by the gravitational field, so that the spaces and times of the theory’s background Minkowski spacetime ceased to be directly measurable. The *dénouement* of the story came in early 1914. It was shown that this conservative path led to the same sort of gravitation theory as did Einstein’s more extravagant speculations on generalizing the principle of relativity. It lead to a theory, akin to general relativity, in which gravitation was incorporated into a dynamical spacetime background. If one abandoned the inaccessible background of Minkowski spacetime and simply assumed that the spacetime of the theory was the one revealed by idealized rod and clock measurements, then it turned out that the gravitation theory was actually the theory of a spacetime that was only conformally flat—gravitation had burst out of special relativity. Most strikingly the theory’s gravitational field equation was an equation strongly reminiscent to modern readers of the field equations of general relativity:

$$R = kT$$

where R is the Riemann curvature scalar and T the trace of the stress-energy tensor. This equation was revealed before Einstein had advanced the generally covariant field equations of general relativity, at a time in which he believed that no such field equations could be physically acceptable.

What makes the story especially interesting are the two leading players other than Nordström. The first was Einstein himself. He was in continued contact with Nordström during the period in which the Nordström theory was developed. We shall see that the theory actually evolved through a continued exchange between them, with Einstein often supplying ideas decisive to the development of the theory. Thus the theory might more accurately be called the "Einstein-Nordström theory." Again it was Einstein in collaboration with Adriaan Fokker who revealed in early 1914 the connection between the theory and conformally flat spacetimes.

The second leading player other than Nordström was not a person but a branch of special relativity, the relativistic mechanics of stressed bodies. This study was under intensive development at this time and had proven to be a locus of remarkably non-classical results. For example it turned out that a moving body would acquire additional energy, inertia and momentum simply by being subjected to stresses, even if the stresses did not elastically deform the body. The latest results of these studies—most notably those of Laue—provided Einstein and Nordström with the means of incorporating the equality of inertial and gravitational mass into their theory. It was also the analysis of stressed bodies within the theory that led directly to the conclusion that even idealized rods and clocks could not measure the background Minkowski spacetime directly but must be affected by the gravitational field. For Einstein and Nordström concluded that a body would also acquire a gravitational mass if subjected to non-deforming stresses and that one had to assume that such a body would alter its size in moving through the gravitational field on pain of violating the law of conservation of energy.

Finally we shall see that the requirement of equality of inertial and gravitational mass is a persistent theme of Einstein's and Nordström's work. However the requirement proves somewhat elastic with both Einstein and Nordström drifting between conflicting versions of it. It will be convenient to prepare the reader by collecting and stating the relevant versions here. On the observational level, the equality could be taken as requiring:

- Uniqueness of free fall:[2] The trajectories of free fall of all bodies are independent of their internal constitution.

Einstein preferred a more restrictive version:

- Independence of vertical acceleration: The vertical acceleration of bodies in free fall is independent of their constitutions and horizontal velocities.

In attempting to devise theories compatible with these observational requirements, Einstein and Nordström considered requiring equality of gravitational mass with

- inertial rest mass
- the inertial mass of closed systems
- the inertial mass of complete static systems
- the inertial mass of a a complete stationary systems[3]

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2 This name is drawn from (Misner et al. 1973, 1050).

3 The notions of complete static and complete stationary systems arise in the context of the mechanics of stressed bodies and are discussed in Sections 9 and 12 below.

More often than not these theoretical requirements failed to bring about the desired observational consequences. Unfortunately it is often unclear precisely which requirement is intended when the equality of inertial and gravitational mass was invoked.

2. THE PROBLEM OF GRAVITATION IMMEDIATELY AFTER 1905

In the years immediately following 1905 it was hard to see that there would be any special problem in modifying Newtonian gravitation theory in order to bring it into accord with the special theory of relativity. The problem was not whether it could be done, but how to choose the best of the many possibilities perceived, given the expectation that relativistic corrections to Newtonian theory might not have measurable consequences even in the very sensitive domain of planetary astronomy. Poincaré (1905, 1507–1508;1906, 166–75), for example, had addressed the problem in his celebrated papers on the dynamics of the electron. He limited himself to seeking an expression for the gravitational force of attraction between two masses that would be Lorentz covariant[4] and would yield the Newtonian limit for bodies at rest. Since this failed to specify a unique result he applied further constraints including the requirement[5] of minimal deviations from Newtonian theory for bodies with small velocities, in order to preserve the Newtonian successes in astronomy. The resulting law, Poincaré noted, was not unique and he indicated how variants consistent with its constraints could be derived by modifying the terms of the original law.

Minkowski (1908, 401–404; 1909, 443–4) also sought a relativistic generalization of the Newtonian expression for the gravitational force acting between two bodies. His analysis was simpler than Poincaré's since merely stating his law in terms of the geometric structures of his four dimensional spacetime was sufficient to guarantee automatic compatibility with special relativity. Where Poincaré (1905, 1508; 1906, 175) had merely noted his expectation that the deviations from Newtonian astronomical prediction introduced by relativistic corrections would be small, Minkowski (1908, 404) computed the deviations due to his law for planetary motions and concluded that they were so small that they allowed no decision to be made concerning the law.

Presumably neither Poincaré nor Minkowski were seeking a fundamental theory of gravitation, for they both considered action-at-a-distance laws at a time when field theories were dominant. Rather the point was to make *plausible*[6] the idea that some slight modification of Newtonian gravitational law was all that was necessary to bring it into accord with special relativity, even if precise determination of that modification was beyond the reach of the current state of observational astronomy.

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4 More precisely he required that the law governing propagation of gravitational action be Lorentz covariant and that the gravitational forces transform in the same way as electromagnetic forces.

5 Also he required that gravitational action propagate forward in time from a given body.

6 The word is Minkowski's. He introduced his treatment of gravitation (Minkowski 1908, 401) with the remark "I would not like to fail to make it plausible that nothing in the phenomena of gravitation can be expected to contradict the assumption of the postulate of relativity."

3. EINSTEIN'S 1907 REJECTION OF LORENTZ COVARIANT GRAVITATION THEORIES

In 1907 Einstein's attention was focussed on the problem of gravitation and relativity theory when he agreed to write a review article on relativity theory for Johannes Stark's *Jahrbuch der Radioaktivität und Elektronik*. The relevant parts of the review article (Einstein 1907a, 414; Section V, 454–62) say nothing of the possibility of a Lorentz covariant gravitation theory. Rather Einstein speculates immediately on the possibility of extending the principle of relativity to accelerated motion. He suggests the relevance of gravitation to this possibility and posits what is later called the principle of equivalence as the first step towards the complete extension of the principle of relativity.

It is only through later reminiscences that we know something of the circumstances leading to these conclusions. The most informative are given over 25 years later in 1933 when Einstein gave a sketch of his pathway to general relativity.[7] In it he wrote (Einstein 1933, 286–87):

> I came a step nearer to the solution of the problem [of extending the principle of relativity] when I attempted to deal with law of gravity within the framework of the special theory of relativity. Like most writers at the time, I tried to frame a *field-law* for gravitation, since it was no longer possible, at least in any natural way, to introduce direct action at a distance owing to the abolition of the notion of absolute simultaneity.
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> The simplest thing was, of course, to retain the Laplacian scalar potential of gravity, and to complete the equation of Poisson in an obvious way by a term differentiated with respect to time in such a way that the special theory of relativity was satisfied. The law of motion of the mass point in a gravitational field had also to be adapted to the special theory of relativity. The path was not so unmistakably marked out here, since the inert mass of a body might depend on the gravitational potential. In fact this was to be expected on account of the principle of the inertia of energy.

While Einstein's verbal description is brief, the type of gravitation theory he alludes to is not too hard to reconstruct. In Newtonian gravitation theory, with a scalar potential ϕ, mass density ρ and G the gravitation constant, the gravitational field equation—the "equation of Poisson"— is[8]

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \frac{\partial^2\phi}{\partial x_i \partial x_i} = 4\pi G\rho. \tag{1}$$

The force **f**, with components (f_1, f_2, f_3), on a point mass m is given by $-m\nabla\phi$ so that

$$f_i = m\frac{dv_i}{dt} = -m\left(\frac{\partial}{\partial x_i}\right)\phi \tag{2}$$

7 A similar account is given more briefly in (Einstein 1949, 58–63).

8 $(x, y, z) = (x_1, x_2, x_3)$ are the usual spatial Cartesian coordinates. The index i ranges over 1, 2, 3. Here and henceforth, summation over repeated indices is implied.

is the law of motion of a point mass m with velocity $v_i = dx_i/dt$ in the gravitational field ϕ.

The adaptation of (1) to special relativity is most straightforward. The added term, differentiated with respect to the time coordinate, converts the Laplacian operator ∇^2 into a Lorentz covariant d' Alembertian $\Box^2$. so that the field equation alluded to by Einstein would be

$$\Box^2\phi = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = -4\pi G\nu. \tag{3}$$

For consistency ϕ is assumed to be Lorentz invariant and the mass density ρ must be replaced with a Lorentz invariant, such as the rest mass density ν used here.

The modification to the law of motion of a point mass is less clear. The natural Lorentz covariant extension of (2) is most obvious if we adopt the four dimensional spacetime methods introduced by Minkowski (1908). Einstein could not have been using these methods in 1907. However I shall write the natural extension here since Einstein gives us little other guide to the form of the equation he considered, since the properties of this equation fit exactly with Einstein's further remarks and since this equation will lead us directly to Nordström's work. The extension of (2) is

$$F_\mu = m\frac{dU_\mu}{d\tau} = -m\frac{\partial\phi}{\partial x_\mu} \tag{4}$$

where F_μ is the four force on a point mass with rest mass m, $U_\mu = dx_\mu/d\tau$ is its four velocity, τ is the proper time and $\mu = 1, 2, 3, 4$.[9] Following the practice of Nordström's papers, the coordinates are $(x_1, x_2, x_3, x_4) = (x, y, z, u = ict)$, for c the speed of light.

Simple as this extension is, it turns out to be incompatible with the kinematics of a Minkowski spacetime. In a Minkowski spacetime, the constancy of c entails that the four velocity U_μ along a world line is orthogonal to the four acceleration $dU_\mu/d\tau$. For we have $c^2 d\tau^2 = -dx_\mu dx_\mu$, so that $c^2 = -U_\mu U_\mu$ and the orthogonality now follows from the constancy of c

$$-\frac{1}{2}\frac{dc^2}{d\tau} = U_\mu\frac{dU_\mu}{d\tau} = 0 \tag{5}$$

(4) and (5) together entail

$$F_\mu U_\mu = -m\frac{\partial\phi}{\partial x_\mu}\frac{dx_\mu}{d\tau} = -m\frac{d\phi}{d\tau} = 0$$

9 Throughout this paper, Latin indices $i, k, \ldots$ range over 1, 2, 3 and Greek indices $\mu, \nu, \ldots$ range over 1, 2, 3, 4.

so that the law (4) can only obtain in a Minkowski spacetime in the extremely narrow case in which the field ϕ is constant along the world line of the particle, i.e.

$$\frac{d\phi}{d\tau} = 0. \tag{6}$$

We shall see below that one escape from this problem published by Nordström involves allowing the rest mass m to be a function of the potential ϕ. Perhaps this is what Einstein referred to above when he noted of the law of motion that the "path was not so unmistakably marked out here, since the inert mass of a body might depend on the gravitational potential."

Whatever the precise form of the modifications Einstein made, he was clearly unhappy with the outcome. Continuing his recollections, he noted:

> These investigations, however, led to a result which raised my strong suspicions. According to classical mechanics, the vertical acceleration of a body in the vertical gravitational field is independent of the horizontal component of its velocity. Hence in such a gravitational field the vertical acceleration of a mechanical system or of its center of gravity works out independently of its internal kinetic energy. But in the theory I advanced, the acceleration of a falling body was not independent of its horizontal velocity or the internal energy of the system.

The result Einstein mentions here is readily recoverable from the law of motion (4) in a special case in which it is compatible with the identity (5). The result has more general applicability, however. The modifications introduced by Nordström to render (4) compatible with (5) vanish in this special case, as would, presumably, other natural modifications that Einstein may have entertained. So this special case is also a special case of these more generally applicable laws.

We consider a coordinate system in which:

(i) the field is time independent $(\partial\phi/\partial t = 0)$ at some event and

(ii) the motion of a point mass m in free fall at that event is such that the "vertical" direction of the field, as given by the acceleration three vector dv_i/dt, is perpendicular to the three velocity v_i, so that

$$v_i \cdot \frac{dv_i}{dt} = 0 \tag{7}$$

and the point's motion is momentarily "horizontal."

Condition (7) greatly simplifies the analysis, since it entails that the t derivative of any function of $v^2 = v_i v_i$ vanishes, so that we have

$$\frac{d}{dt}\frac{dt}{d\tau} = \frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) = 0. \tag{8}$$

Notice also that in this case (7) entails that $v_i \frac{\partial\phi}{\partial x_i} = 0$ so that

$$\frac{d\phi}{d\tau} = \frac{dt}{d\tau}\left(\frac{\partial\phi}{\partial t} + v_i \frac{\partial\phi}{\partial x_i}\right) = 0$$

and (6) and then also (5) are satisfied for this special case. Finally an expression for the acceleration of the point mass now follows directly from (4) and is[10]

$$\frac{dv_i}{dt} = -\left(1 - \frac{v^2}{c^2}\right)\frac{\partial\phi}{\partial x_i}. \tag{9}$$

According to (9), the greater the horizontal velocity v, the less the vertical acceleration, so that this acceleration is dependent on the horizontal velocity as Einstein claimed.

Einstein also claims in his remarks that the vertical acceleration would not be independent of the internal energy of the falling system. This result is suggested by equation (9), which tells us that the vertical acceleration of a point mass diminishes with its kinetic energy if the velocity generating that kinetic energy is horizontally directed. If we apply this result to the particles of a kinetic gas, we infer that in general each individual particle will fall slower the greater its velocity. Presumably this result applies to the whole system of a kinetic gas so that the gas falls slower the greater the kinetic energy of its particles, that is, the greater its internal energy. This example of a kinetic gas was precisely the one given by Einstein in an informal lecture on April 14, 1954 in Princeton according to lecture notes taken by J. A. Wheeler.[11]

Einstein continued his recollections by explaining that he felt these results so contradicted experience that he abandoned the search for a Lorentz covariant gravitation theory.

> This did not fit with the old experimental fact that all bodies have the same acceleration in a gravitational field. This law, which may also be formulated as the law of the equality of inertial and gravitational mass, was now brought home to me in all its significance. I was in the highest degree amazed at its existence and guessed that in it must lie the key to a deeper understanding of inertia and gravitation. I had no serious doubts about its strict validity even without knowing the results of the admirable experiments of Eötvos, which—if my memory is right—I only came to know later. I now abandoned as inadequate the attempt to treat the problem of gravitation, in the manner outlined above, within the framework of the special theory of relativity. It clearly failed to do justice to the most fundamental property of gravitation.

Einstein then recounted briefly the introduction of the principle of equivalence, upon which would be based his continued work on gravitation and relativity, and concluded

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10 Since $\frac{dU_i}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt}\left(v_i \frac{dt}{d\tau}\right)$, (9) follows directly from (4) using (8).

11 Wheeler's notes read "I had to write a paper about the content of special relativity. Then I came to the question how to handle gravity. The object falls with a different acceleration if it is moving than if it is not moving. ... Thus a gas falls with another acceleration if heated than if not heated. I felt this is not true ... " (Wheeler 1979, 188).

> Such reflections kept me busy from 1908 to 1911, and I attempted to draw special conclusions from them, of which I do not propose to speak here. For the moment the one important thing was the discovery that a reasonable theory of gravitation could only be hoped for from an extension of the principle of relativity.

Our sources concerning Einstein's 1907 renunciation of Lorentz covariant gravitation theories are largely later recollections so we should be somewhat wary of them. Nonetheless they all agree in the essential details:[12] Einstein began his attempts to discover a Lorentz covariant theory of gravitation as a part of his work on his 1907 *Jahrbuch* review article. He found an inconsistency between these attempts and the exact equality of inertial and gravitational mass, which he found sufficiently disturbing to lead him to abandon the search for such theories.

We shall see shortly that Einstein's 1907 evaluation and dismissal of the prospects of a Lorentz covariant gravitation theory—as reconstructed above—was far too hasty. Within a few years Einstein himself would play a role in showing that one could construct a Lorentz covariant gravitation theory that was fully compatible with the exact equality of inertial and gravitational mass. We can understand why Einstein's 1907 analysis would be hurried, however, once we realize that the he could have devoted very little time to contemplation of the prospects of a Lorentz covariant gravitation theory. He accepted the commission of the *Jahrbuch*'s editor, Stark, to write the review in a letter of September 25, 1907 (EA 22 333) and the lengthy and completed article was submitted to the journal on December 4, 1907, a little over two months later. This period must have been a very busy one for Einstein. As he explained to Stark in the September 25 letter, he was not well read in the current literature pertinent to relativity theory, since the library was closed during his free time. He asked Stark to send him relevant publications that he might not have seen.[13] During this period, whatever time Einstein could have spent privately contemplating the prospects of a Lorentz covariant gravitation theory would have been multiply diluted. There were the attractions of the principle of equivalence, whose advent so dazzled him that he called it the "happiest thought of [his] life".[14] Its exploitation attracted all the pages of the review article which concern gravitation and in which the prospects of a Lorentz covariant gravitation theory are not even mentioned. Further diluting his time would be the demands of the remaining sections of the review article. The section devoted to gravitation filled only nine of the article's fifty two pages. Finally, of

12 See also the 1920 recollections of Einstein on p. 23, "Grundgedanken und Methoden der Relativitätstheorie in ihrer Entwicklung dargestellt," unpublished manuscript, control number 2 070, Duplicate Einstein Archive, Mudd Manuscript Library, Princeton, NJ. (Henceforth "EA 2 070".) Einstein recalls:
"When, in the year 1907, I was working on a summary essay concerning the special theory of relativity for the *Jahrbuch für Radioaktivität und Elektronik* [sic], I had to try to modify Newton's theory of gravitation in such a way that it would fit into the theory [of relativity]. Attempts in this direction showed the possibility of carrying out this enterprise, but they did not satisfy me because they had to be supported by hypotheses without physical basis." Translation from (Holton 1975, 369–71).

13 Einstein thanked him for sending papers in a letter of October 4, 1907 (EA, 22 320).

14 In Einstein's 1920 manuscript (EA 2 070, 23–25).

course, there were the obligations of his job at the patent office. It is no wonder that he lamented to Stark in a letter of November 1, 1907, that he worked on the article in his "unfortunately truly meagerly measured free time" (EA, 22 335).

4. EINSTEIN'S ARGUMENT OF JULY 1912

If the Einstein of 1907 had not probed deeply the prospects of Lorentz covariant gravitation theories, we might well wonder if he returned to give the problem more thorough treatment in the years following. We have good reason to believe that as late as July 1912, Einstein had made no significant advance on his deliberations of 1907.[15] Our source is an acrimonious dispute raging at this time between Einstein and Max Abraham. In language that rarely appeared in the unpolluted pages of *Annalen der Physik*, Abraham (1912c, 1056) accused Einstein's theory of relativity of having "exerted an hypnotic influence especially on the youngest mathematical physicists which threatened to hamper the healthy development of theoretical physics." He rejoiced especially in what he saw as major retractions in Einstein's latest papers on relativity and gravitation. Einstein (1911) involved a theory of gravitation which gave up the constancy of the velocity of light and Einstein (1912a, 1912b) even dispensed with the requirement of the invariance of the equations of motion under Lorentz transformation. These concessions, concluded Abraham triumphantly, were the "death blow" for relativity theory.

Einstein took this attack very seriously. His correspondence from this time, a simple gauge of the focus of his thoughts, was filled with remarks on Abraham. He repeatedly condemned Abraham's (1912a,1912b) new theory of gravitation, which had adopted Einstein's idea of a variable speed of light as the gravitational potential. "A stately beast that lacks three legs," he wrote scathingly of the theory to Ludwig Hopf.[16] He anticipated the dispute with Abraham with some relish, writing to Hopf earlier of the coming "difficult ink duel."[17] The public dispute ended fairly quickly, however, with Einstein publishing a measured and detailed reply (Einstein 1912d) and then refusing to reply to Abraham's rejoinder (Abraham 1912d). Instead Einstein published a short note (Einstein 1912e) indicating that both parties had stated their views and asking readers not to interpret Einstein's silence as agreement. Nonetheless Einstein continued to hold a high opinion of Abraham as a physicist, lamenting in a letter to Hopf that Abraham's theory was "truly superficial, contrary to his [Abraham's] usual practice."[18]

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15 This is a little surprising. Einstein had neglected gravitation in 1908–1911, possibly because of his preoccupation with the problem of quanta. (See Pais (1983, 187–90.)) However he had returned to gravitation with vigor with his June 1911 submission of Einstein (1911) and by July 1912, the time of his dispute with Abraham, he had completed at least two more novel papers on the subject, (Einstein 1912a, 1912b), and possibly a third, (Einstein 1912c).

16 Einstein to Ludwig Hopf, 16 August 1912, (EA 13 288).

17 Einstein to Ludwig Hopf, December 1911 (?), (EA 13 282).

Under these circumstances, Einstein had every incentive to make the best case for his new work on gravitation. In particular, we would expect Einstein to advance the best arguments available to him to justify his 1907 judgement of the untenability of Lorentz covariant gravitation theories, for it was this conclusion that necessitated the consideration of gravitation theories that went beyond special relativity. What he included in his response shows us that as late as July 4, 1912—the date of submission of his response (Einstein 1912d)—his grounds for this judgement had advanced very little beyond those he recalled having in 1907. He wrote (pp. 1062–63)

> One of the most important results of the theory of relativity is the realization that every energy E possesses an inertia (E/c^2) proportional to it. Since each inertial mass is at the same time a gravitational mass, as far as our experience goes, we cannot help but ascribe to each energy E a gravitational mass E/c^2.[19] From this it follows immediately that gravitation acts more strongly on a moving body than on the same body in case it is at rest.
>
> If the gravitational field is to be interpreted in the sense of our current theory of relativity, this can happen only in two ways. One can conceive of the gravitation vector either as a four-vector or a six-vector. For each of these two cases there are transformation formulae for the transition to a uniformly moving reference system. By means of these transformation formulae and the transformation formulae for ponderomotive forces one can find for both cases the forces acting on moving material points in a static gravitational field. However from this one arrives at results which conflict with the consequences mentioned of the law of the gravitational mass of energy. Therefore it seems that the gravitation vector cannot be incorporated without contradiction in the scheme of the current theory of relativity.

Einstein's argument is a fairly minor embellishment of the reflections summarized in Section 3 above. Einstein has replaced a single theory, embodied in equations such as (3) and (4), with two general classes of gravitation theory, the four-vector and six-vector theory. In both classes of gravitation theory, in the case of moving masses, Einstein claims that the gravitational field fails to act on them in proportion to their total energy, in effect violating the requirement of equality of inertial and gravitational mass.

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18 Einstein to Ludwig Hopf, 12 June 1912, (EA 13 286). Einstein retained his high opinion of Abraham as a physicist. Late the following year, after his work had advanced into the first sketch of the general theory of relativity, Einstein conceded to his confidant Besso that "Abraham has the most understanding [of the new theory]." Einstein to Michele Besso, end of 1913, in (Speziali 1972, 50). For further mention of Abraham in correspondence from this period see Einstein to Heinrich Zangger, 27 January 1912, (EA 39 644); Einstein to Wilhelm Wien, 27 January 1912, (EA 23 548); Einstein to Heinrich Zangger, 29 February 1912, (EA 39 653); Einstein to Wilhelm Wien, 24 February 1912, (EA 23 550); Einstein to Heinrich Zangger, 20 May 1912, (EA 39 655); Einstein to Michele Besso, 26 March 1912, (EA 7 066); Einstein to Heinrich Zangger, summer 1912, (EA 39 657); Einstein to Arnold Sommerfeld, 29 October 1912, (EA 21 380). See also (Pais 1982, 231–32).

19 At this point, Einstein inserts the footnote:
Hr. Langevin has orally called my attention to the fact that one comes to a contradiction with experience if one does not make this assumption. That is, in radioactive decay large quantities of energy are given off, so that the *inertial* mass of the matter must diminish. If the gravitational mass were not to diminish proportionally, then the gravitational acceleration of bodies made out of different elements would have to be demonstrably different in the same gravitational field.

Einstein does not give a full derivation of the result claimed. However we can reconstruct what he intended from the derivation sketch given. The two types of force fields correspond to the "spacetime vectors type I and II" introduced by Minkowski (1908, § 5), which soon came to be known as four- and six-vector fields, respectively (Sommerfeld 1910, 750). They represented the two types of force fields then examined routinely in physics. The four-vector corresponds to the modern vector of a four dimensional manifold. The gravitational four-force F_μ acting on a body with rest mass m in a four-vector theory is

$$F_\mu = mG_\mu. \tag{10a}$$

An example of such a theory is given by (4) above in which the gravitation four-vector G_μ is set equal to $-\partial\phi/\partial x_\mu$. The six-vector corresponds to our modern antisymmetric second rank tensor which has six independent components. The classic example of a six-vector is what Sommerfeld called "the six-vector ... of the electromagnetic field" (Sommerfeld 1910, 754). We would now identify it as the Maxwell field tensor. Presumably Einstein intended a six-vector gravitation theory to be modelled after electrodynamics, so that the gravitational four-force F_μ acting on a body with rest mass m and four-velocity U_ν in such a theory would be given by

$$F_\mu = mG_{\mu\nu}U_\nu. \tag{11a}$$

The gravitation six-vector, $G_{\mu\nu}$, satisfies the antisymmetry condition $G_{\mu\nu} = -G_{\nu\mu}$. This antisymmetry guarantees compatibility with the identity (5) since it forces $F_\mu U_\mu = 0$.

Einstein claims that one needs only the transformation formulae for four and six-vectors and for ponderomotive forces. to arrive at the results. However, since both (10a) and (11a) are Lorentz covariant, application of the transformation formulae to these equations simply returns equations of identical form—an uninformative outcome. We do recover results of the type Einstein claims, however, if we apply these transformation formulae to non-covariant specializations of (10a) and (11a).

We consider arbitrary four and six-vector gravitational fields G_μ and $G_{\mu\nu}$. In each there is a body of mass m in free fall. In each case, select and orient a coordinate system $S'(x', y', z', u' = ict')$ in such a way that each mass is instantaneously at rest and is accelerating only in $y' = x'_2$ direction. For these coordinate systems, the three spatial components of the four-force, F'_i , are equal to the three components of the three force, f'_i, acting on the masses. In particular the $x'_2 = y'$ component f'_2 of the three force is given in each case by

$$f'_2 = mG'_2, \tag{10b}$$

$$f'_2 = mG'_{2\nu}U'_\nu = mG'_{24}\,ic \tag{11b}$$

since $U'_\mu = (0, 0, 0, ic)$. If we now transform from S' to a reference system $S(x, y, z, u = ict)$ moving at velocity v in the $x'_1 = x'$ direction S', then the relevant Lorentz transformation formulae are

$$f'_2 = \frac{f_2}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad G'_{24} = \frac{G_{24}-(iv/c)G_{21}}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Thus far we have not restricted the choice of four or six-vector fields G_μ and $G_{\mu\nu}$. The considerations that follow are simplified if we consider a special case of the six-vector field $G_{\mu\nu}$ in which $G_{21} = 0$.[20] Substituting with these transformation formulae for f'_2 and G'_{24} in this special in (10b) and (11b), we recover

$$f_2 = m\sqrt{1-\frac{v^2}{c^2}}G_2 \tag{10c}$$

$$f_2 = mG_{24}ic. \tag{11c}$$

These two equations describe the component of gravitational three-force, f_2, in the "vertical" $x_2 = y$ direction on a mass m moving with velocity v in the "horizontal" $x_1 = x$ direction. In his 1912 argument, Einstein noted that the inertia of energy and the equality of inertial and gravitational mass leads us to expect that "gravitation acts more strongly on a moving body than on the same body in case it is at rest." We read directly from equations (10c) and (11c) that both four and six-vector theories fail to satisfy this condition. The gravitational force is independent of velocity in the six-vector case and actually decreases with velocity in the four-vector case. To meet Einstein's requirements, the gravitational force would need to increase with velocity, in direct proportion to the mass's energy $mc^2/\sqrt{1-(v^2/c^2)}$.

We can also confirm that (10c) and (11c) lead to the result that the vertical acceleration of the masses is not independent of their horizontal velocities. To see this, note that, were the masses of (10c) and (11c) instantaneously at rest, the vertical forces exerted by the two fields would be respectively

$$f_2^{\text{rest}} = mG_2, \quad f_2^{\text{rest}} = mG_{24}ic.$$

In all cases, the three velocity and three-accelerations are perpendicular, so that condition (8) holds. Therefore we have

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20 This restriction does not compromise the generality of Einstein's claim. If a Lorentz covariant theory proves inadequate in a special case, that is sufficient to demonstrate its general inadequacy. A natural instance of a six-vector field $G_{\mu\nu}$ in which $G_{21} = 0$ is easy to construct. Following the model of electromagnetism, we assume that $G_{\mu\nu}$ is generated by a vector potential A_μ, according to

$$G_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}.$$

We choose a "gravito-static" field in $S(x, y, z, u)$, that is, one that is analogous to the electrostatic field, by setting $A_\mu = (0, 0, 0, A_4)$. Since A_1 and A_2 are everywhere vanishing, $G_{21} = 0$. Finally note that Einstein does explicitly restrict his 1912 claim to static gravitational fields. Perhaps he also considered simplifying special examples of this type.

$$f_i = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} m \frac{dv_i}{dt}, \quad f_i^{\text{rest}} = m\left(\frac{dv_i}{dt}\right)^{\text{rest}}.$$

Combining these results with (10c) and (11c) we recover expressions for the vertical acceleration dv_2/dt of the masses in terms of the acceleration $(dv_2/dt)^{\text{rest}}$ they would have had if they had no horizontal velocity

$$\frac{dv_2}{dt} = \left(1-\frac{v^2}{c^2}\right)\left(\frac{dv_2}{dt}\right)^{\text{rest}} \tag{10d}$$

$$\frac{dv_2}{dt} = \sqrt{1-\frac{v^2}{c^2}}\left(\frac{dv_2}{dt}\right)^{\text{rest}} \tag{11d}$$

We see that in both four and six-vector cases the vertical acceleration decreases with horizontal velocity, with equation (10d) generalizing the result in equation (9).

5. A GRAVITATION THEORY MODELLED AFTER MAXWELL'S ELECTROMAGNETISM?

Einstein's mention of a six-vector theory of gravitation in his 1912 response to Abraham raises the question of Einstein's attitude to a very obvious strategy of relativization of Newtonian gravitation theory. With hindsight one can view the transition from the theory of Coulomb electrostatic fields to full Maxwell electromagnetism as the first successful relativization of a field theory. Now Newtonian gravitation theory is formally identical to the theory of electrostatic fields excepting a change of sign needed to ensure that gravitational masses attract where like electric charges repel. This suggests that one can relativize Newtonian gravitation theory by augmenting it to a theory formally identical to Maxwell theory excepting this same change of sign.

While it is only with hindsight that one sees the transition from electrostatics to electromagnetism as a relativization, Einstein had certainly developed this hindsight by 1913. In his (Einstein 1913, 1250) he noted that Newtonian theory has sufficed so far for celestial mechanics because of the smallness of the speeds and accelerations of the heavenly bodies. Were these motions to be governed instead by electric forces of similar magnitude, one would need only Coulomb' s law to calculate these motions with great accuracy. Maxwell's theory would not be required. The problem of relativizing gravitation theory, Einstein continued, corresponded exactly to this problem: if we knew only experimentally of electrostatics but that electrical action could not propagate faster than light, would we be able to develop Maxwell electromagnetics? In the same paper Einstein proceeded to show (p. 1261) that his early 1913 version of general relativity reduced in suitable weak field approximation to a theory with a four-vector field potential that was formally analogous to electrodynamics. It was this approximation that yielded the weak field effects we now label as "Machian." The previous year, when seeking similar effects in his 1912 theory of static gravitational fields, Einstein demonstrated that he then expected a relativized gravitation theory to be formally analogous to electrodynamics at some level. For then he wrote a paper with the revealing title "Is there a gravitational effect that is analogous to electrodynamic induction?" (Einstein 1912c).

The celebrated defect of a theory of gravitation modelled after Maxwell electromagnetism was first pointed out by Maxwell himself (Maxwell 1864, 571). In such a theory, due to the change of signs, the energy density of the gravitational field is negative and becomes more negative as the field becomes stronger. In order not to introduce net negative energies into the theory, one must then suppose that space, in the absence of gravitational forces, must contain a positive energy density sufficiently great to offset the negative energy of any possible field strength. Maxwell professed himself baffled by the question of how a medium could possess such properties and renounced further work on the problem. As it turns out it was Einstein's foe, Abraham, shortly after his exchange with Einstein, who refined Maxwell's concern into a more telling objection. In a lecture of October 19, 1912, he reviewed his own gravitation theory based on Einstein's idea of using the speed of light as a gravitational potential. (Abraham 1912e) He first reflected (pp. 193–94), however, on a gravitation theory modelled after Maxwell electromagnetism. In such a theory, a mass, set into oscillation, would emit waves analogous to light waves. However, because of the change of sign, the energy flow would not be away from the mass but towards it, so that the energy of oscillation would increase. In other words such an oscillating mass would have no stable equilibrium. Similar difficulties were reported by him for gravitation theories of Maxwellian form due to H.A. Lorentz and R. Gans.

What was Einstein's attitude to such a theory of gravitation? He was clearly aware of the formal possibility of such a theory in 1912 and 1913. From his failure to exploit such a theory, we can only assume that he did not think it an adequate means of relativizing gravitation.[21] Unfortunately I know of no source from that period through which Einstein states a definite view on the matter beyond the brief remarks in his exchange with Abraham. We shall see that Einstein is about to renounce the conclusion of his reply to Abraham, that a Lorentz covariant theory cannot capture the equality of inertial and gravitational mass, at least for the case of Nordström's theory of gravitation. Did Einstein have other reservations about six-vector theories of gravitation? How seriously, for example, did he regard the negative field energy problem in such a theory?

The idea of an analogy between a relativized gravitation theory and electrodynamics seems to play no significant role in the methods Einstein used to generate relativized gravitation theories. The effect analogous to electrodynamic induction of

21 Notice these reservations must have amounted to more than the observation that such a theory fails to extend the relativity of motion to acceleration. In (Einstein 1913), immediately after his remarks on the similarity between the problems of relativizing gravitation and electrostatics, he considers Lorentz covariant gravitation theories. The only theory taken seriously in this category is a version of Nordström's theory of gravitation.

(Einstein 1912c), for example, was derived fully within Einstein's 1912 theory of static gravitational fields and the analogy to electrodynamics appeared only in the description of the final result. In general, the mention of an analogy to electrodynamics seems intended solely to aid Einstein's readers in understanding the enterprise and physical effects appearing in the relativized theories of gravitation by relating them to an example familiar to his readers. That we have any surviving, written remarks by Einstein directly on this matter we owe to J.W. Killian. Some thirty years later, in a letter of June 9, 1943 (EA 14 261) to Einstein, Killian proposed a gravitation theory modelled after Maxwell electromagnetism.[22] Einstein's reply of June 28, 1943, gives a fairly thorough statement of his attitude at that time to this theory.[23]

Because there was no question of experimental support for the theory, Einstein proposed to speak only to its formal properties. To begin, he noted, Maxwell's equations only form a complete theory for parts of space free of source charges, for the theory cannot determine the velocity field of the charge distribution without further assumption. After Lorentz, to form a complete theory, it was assumed that charges were carried by ponderable masses whose motions followed from Newton's laws. What Einstein called "real difficulties" arise only in explaining inertia. These difficulties result from the negative gravitational field energy density in the theory. Assuming, apparently, that the energy of a mass in the theory would reside in its gravitational field, Einstein pointed out that the kinetic energy of a moving mass point would be negative. This negativity would have to be overcome by a device entirely arbitrary from the perspective of the theory's equations, the introduction of a compensating positive energy density located within the masses. This difficulty is more serious for the gravitational version of the theory, for, in the electromagnetic theory, the positivity of electromagnetic field energy density allows one to locate all the energy of a charge in its electromagnetic field.

Calling the preceding difficulty "the fundamental problem of the wrong sign," Einstein closed his letter with brief treatment of two further and, by suggestion, lesser difficulties. The proposed theory could not account, Einstein continued, for the proportionality of inertial and gravitational mass. Here we finally see the concern that drove Einstein's work on Lorentz covariant gravitation theory in the decade following 1907. Yet Einstein does not use the transformation arguments of this early period to establish the failure of the proposed theory to yield this proportionality. Instead he continues to imagine that the energy and therefore inertia of a mass resides in its gravitational field. Some fixed quantity of gravitational mass could be configured in many different ways. Thus it follows that the one quantity of gravitational mass could be associated with many different gravitational fields and thus many different inertial masses, in contradiction with the proportionality sought.[24] Finally Einstein remarked

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22 Einstein addresses his reply to "Mr. J.W. Killian, Dept. of Physics, Rockerfeller Hall, Ithaca N.Y."

23 EA 14 265 is an autograph draft of the letter in German. EA 14 264 is an unsigned typescript of the English translation. There are some significant differences of content between the two, indicating further editing of content presumably by Einstein between the draft and typescript.

that the proposed theory allows no interaction between electromagnetic and gravitational fields other than through charged, ponderable masses. Thus it could not explain the bending of starlight in a gravitational field.

6. NORDSTRÖM'S FIRST THEORY OF GRAVITATION

The dispute between Einstein and Abraham was observed with interest by a Finnish physicist, Gunnar Nordström.[25] In a paper submitted to *Physikalische Zeitschrift* in October 1912 (Nordström 1912), he explained that Einstein's hypothesis that the speed of light c depends on the gravitational potential led to considerable problems such as revealed in the Einstein-Abraham dispute. Nordström announced (p. 1126) that he believed he had found an alternative to Einstein's hypothesis which would

> ... leave c constant and still adapt the theory of gravitation to the relativity principle in such a way that gravitational and inertial masses are equal.

The theory of gravitation which Nordström developed was a slight modification of the theory embodied in equations (3) and (4) above. Selecting the commonly used coordinates $x, y, z = ict$, Nordström gave his version of the field equation (3):

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} + \frac{\partial^2\Phi}{\partial u^2} = 4\pi f\gamma \tag{12}$$

where Φ is the gravitational potential, γ the rest density of matter and f the gravitational constant. As he noted, this field equation was identical to the one advanced by Abraham (1912a, equation (1)) in the latter's gravitation theory.

Where Nordström differed from Abraham, however, was in the treatment of the force equation (4). This equation, as Nordström pointed out, is incompatible with the constancy of c. We saw above that force equation (4), in conjunction with the constancy of c in equation (5) entails the unphysical condition (6). Abraham had resolved the problem by invoking Einstein's hypothesis that c not be constant but vary with gravitational potential so that condition (5) no longer obtains. Thus Abraham's gravitation theory was no longer a special relativistic theory. Nordström, determined to preserve special relativity and the constancy of c, offered a choice of two modified versions of (4).

First, one could allow the rest mass m of a body in a gravitational field to vary with gravitational potential. Defining

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24 In the German autograph draft (EA 14 265), Einstein imagines some fixed quantity of gravitational mass distributed between two bodies. The field strength they generate, and therefore their energy and inertial mass, would increase as the bodies were concentrated into smaller regions of space. (We may conjecture here that Einstein is ignoring the fact the field energy becomes more *negative* as the field strength increases.) The translated typescript (EA 14 264) simplifies the example by imagining that the gravitational mass is located in a single corpuscle, whose field and thence inertia varies with the radius of the corpuscle.

25 For a brief account of Nordström's life and his contribution to gravitation theory see (Isaksson 1985).

$$\mathfrak{F}_\mu = -\frac{\partial \Phi}{\partial x_\mu},$$

the four force on a body of mass m is

$$m\mathfrak{F}_\mu = -m\frac{\partial \Phi}{\partial x_\mu} = \frac{d}{d\tau}(m\mathfrak{a}_\mu) = m\frac{d\mathfrak{a}_\mu}{d\tau} + \mathfrak{a}_\mu\frac{dm}{d\tau} \tag{13}$$

where $\mathfrak{a}_\mu$ is the mass' four velocity and τ proper time.[26] The dependence of m on Φ introduces the additional, final term in $dm/d\tau$, which prevents the derivation of the disastrous condition (6). In its place, by contracting (13) with $\mathfrak{a}_\mu = \frac{dx_\mu}{dt}$ and noting that $\mathfrak{a}_\mu\mathfrak{a}_\mu = -c^2$, Nordström recovered the condition

$$m\frac{d\Phi}{d\tau} = c^2\frac{dm}{d\tau} \tag{14}$$

which yields an expression for the Φ dependence of m upon integration

$$m = m_0\exp\left(\frac{\Phi}{c^2}\right) \tag{15}$$

where m_0 is the value of m when $\Phi = 0$. Using (14) to substitute $\frac{dm}{d\tau}$ in (13), Nordström then recovered an equation of motion for a mass point independent of m

$$-\frac{\partial \Phi}{\partial x_\mu} = \frac{d\mathfrak{a}_\mu}{d\tau} + \frac{\mathfrak{a}_\mu}{c^2}\frac{d\Phi}{d\tau}. \tag{16}$$

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26 *Note on notation*: The notation used in the sequence of papers discussed here varies. I shall follow the notation of the original papers as it changes, with one exception for brevity. Where the components of an equation such as (13) were written out explicitly as four equations

$$-m\frac{\partial \Phi}{\partial x} = \frac{d}{d\tau}(m\mathfrak{a}_x) = m\frac{d\mathfrak{a}_x}{d\tau} + \mathfrak{a}_x\frac{dm}{d\tau},$$

$$-m\frac{\partial \Phi}{\partial y} = \frac{d}{d\tau}(m\mathfrak{a}_y) = m\frac{d\mathfrak{a}_y}{d\tau} + \mathfrak{a}_y\frac{dm}{\tau},$$

etc.

I silently introduce the coordinates $x_\mu = (x_1, x_2, x_3, x_4) = (x, y, z, u) = ict$ and corresponding index notation as in equation (13) above.

Nordström's second alternative to force equation (4) preserved the independence of m from the potential. The quantity $m\mathfrak{F}_\mu = -m\frac{\partial\Phi}{\partial x_\mu}$ could not be set equal to the gravitational four-force on a mass $m, \frac{d}{d\tau}(m\mathfrak{a}_\mu)$, for that would be incompatible with the orthogonality (5) of four velocity and four acceleration. However one can retain compatibility with this orthogonality if one selects as the four force only that part of $m\mathfrak{F}_\mu$ which is orthogonal to the four-velocity $\mathfrak{a}_\mu$. This yields the second alternative for the force equation

$$\frac{d}{d\tau}(m\mathfrak{a}_\mu) = m\mathfrak{F}_\mu + m\frac{\mathfrak{a}_\mu}{c^2}(\mathfrak{F}_\nu\mathfrak{a}_\nu) = \begin{pmatrix}\text{part of } m\mathfrak{F}_\mu \\ \text{orthogonal} \\ \text{to } \mathfrak{a}_\mu\end{pmatrix}, \tag{17}$$

Nordström somewhat casually noted that he would use the first alternative, since it corresponded to "the position of most researchers in the domain of relativity theory." (p. 1126) Indeed Nordström proceeded to show that both force equations lead to exactly the same equation of motion (16) for a point mass, planting the suggestion that the choice between alternatives could be made arbitrarily.

Regular readers of *Physikalische Zeitschrift*, however, would know that Nordström's decision between the two alternatives could not have been made so casually by him. For in late 1909 and early 1910, Nordström had engaged in a lively public dispute with none other than Abraham on a problem in relativistic electrodynamics that was in formal terms virtually the twin of the choice between the force laws (13) and (17). (Nordström 1909, 1910; Abraham 1909, 1910.) The problem centered on the correct expression for the four force density on a matter distribution in the case of Joule heating. The usual formula for the four force density $\mathfrak{K}_\mu$ on a mass distribution with rest mass density ν is, in the notation of Abraham (1910),

$$\mathfrak{K}_\mu = \nu\frac{d}{d\tau}\left(\frac{dx_\mu}{d\tau}\right)$$

with τ proper time and coordinates $x_\mu = (x, y, z, u = ict)$. In the case of Joule heating, it turns out that this expression leads to a contradiction with the orthogonality condition (5). The two escapes from this problem at issue in the dispute are formally the same as the two alternative gravitational force laws. Nordström defended Minkowski's approach, which took the four force density to be that part of $\mathfrak{K}_\mu$ orthogonal to the matter four velocity $\left(\frac{dx_\mu}{d\tau}\right)$—the counterpart of force law (17).

Abraham concluded that the rest mass density ν would increases in response to the energy of Joule heat generated. He showed a consistent system could be achieved if one now imported this variable ν into the scope of the $d/d\tau$ operator in the expression for $\mathfrak{K}_\mu$—an escape that is the counterpart of (13). Abraham's escape was judged the only tenable one when he was able to show that it yielded the then standard Lorentz transformation formula for heat whereas the Nordström-Minkowski formula did not.[27]

The connection between Nordström's 1912 gravitation theory and this earlier dispute surfaced only in extremely abbreviated form in Nordström (1912). In a brief sentence in the body of the paper, Nordström noted gingerly that (p. 1127)

> the latter way of thinking [alternative (17)] corresponds to Minkowski's original, that treated first [alternative (13)] to that held by Laue and Abraham.

That Nordström had any stake in the differing viewpoints is only revealed in a footnote to this sentence in which the reader is invited to consult "the discussion between Abraham and the author," followed by a citation to the four papers forming the dispute. Nordström then closes with the remark[28]

> I now take the position then taken up by Abraham.

Nordström now continued his treatment of gravitation by extending the discussion from isolated point masses to the case of continuous matter distributions—an area in which he had some interest and expertise (Nordström 1911). He derived a series of results in a straightforward manner. They included expressions for gravitational four force density on a continuous mass distributions and the corresponding equations of motion, expressions for the energy density and flux due to both gravitational field and matter distribution, the gravitational field stress-energy tensor and the laws of conservation of energy and momentum.

The last result Nordström derives concerns point masses. He notes that the field equation (12) admits the familiar retarded potential as a solution for a matter distribution with rest density γ

$$\Phi(x_0, y_0, z_0, t) = -f\int\frac{dxdydz}{r}\gamma_{t-r/c} + \text{constant}$$

where $\gamma_{t-r/c}$ is γ evaluated at time $t-r/c$, the integration extends over all of three dimensional space and $r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$. It follows from the factor of $1/r$ in the integral that the potential Φ at a true point mass would be $-\infty$. Allowing for the dependence of mass on potential given in (15), it follows

27 Thus the authoritative judgement of Pauli's Teubner Encyklopädie article (Pauli 1921, 108) is that "Nordström's objections cannot be upheld." For a lengthy discussion of this debate and an indication that the issues were not so simple, see (Liu 1991).

28 His earlier work (Nordström 1911) had explicitly employed Abraham's "force concept," although Nordström had then noted very evasively that, in using it, he "wish[es] to assert no definite opinion on the correctness of one or other of the two concepts" (p. 854).

that the mass of such a point would have to be zero so that true point masses cannot exist. Nordström concluded with confidence, however, that he could see no contradictions arising from this result.

We may wonder at Nordström's lack of concern over this result. It would be thoroughly intelligible, however, if Nordström were to agree—as Nordström's later (Nordström 1913a, 856) suggests —with Laue's view on the relation between the theory of point masses and of continua. Laue had urged that the former ought to be derived from the latter (Laue 1911a, 525). Under this view, the properties of extended masses are derived from consideration of discrete volumes in a continuous matter distribution, not from the accumulated behavior of many point masses. So the impossibility of point masses in Nordström's theory would present no obstacle in his generation of the behavior of extended bodies.

7. EINSTEIN REPLIES

In advancing his theory, Nordström had claimed to do precisely what Einstein had claimed impossible: the construction of a Lorentz covariant theory of gravitation in which the equality of inertial and gravitational mass held. We need not guess whether Einstein communicated his displeasure to Nordström, for Einstein's missive was sufficiently swift for Nordström to acknowledge it in an addendum (p. 1129) to his paper which read

> *Addendum to proofs*. From a letter from Herr Prof. Dr. A. Einstein I learn that he had already earlier concerned himself with the possibility used above by me for treating gravitational phenomena in a simple way. He however came to the conviction that the consequences of such a theory cannot correspond with reality. In a simple example he shows that, according to this theory, a rotating system in a gravitational field will acquire a smaller acceleration than a non-rotating system.

Einstein's objection to Nordström is clearly an instance of his then standard objection to Lorentz covariant theories of gravitation: in such theories the acceleration of fall is not independent of a body's energy so that the equality of inertial and gravitational mass is violated. It is not hard to guess how Einstein would establish this result for a spinning body in Nordström's theory. It would seem to follow directly from the familiar equation (9) which holds in Nordström's theory and which says, loosely speaking, that a body falls slower if it has a greater horizontal velocity. Indeed, as we shall see below, this is precisely how Nordström shortly establishes the result in his next paper on gravitation theory.

Nordström continued and completed his addendum with a somewhat casual dismissal of Einstein's objection.

> I do not find this result dubious in itself, for the difference is too small to yield a contradiction with experience. Of course, the result under discussion shows that my theory is not compatible with Einstein's principle of equivalence, according to which an unaccelerated reference system in a homogeneous gravitational field is equivalent to an accelerated reference system in a gravitation free space.

> In this circumstance, however, I do not see a sufficient reason to reject the theory. For, even though Einstein's hypothesis is extraordinarily ingenious, on the other hand it still provides great difficulties. Therefore other attempts at treating gravitation are also desirable and I want to provide a contribution to them with my communication.

Nordström's reply is thoroughly reasonable. The requirement of exact equality of inertial and gravitational mass was clearly an obsession of Einstein's thinking at this time and not shared by Einstein's contemporaries. The now celebrated Eötvos experiment had not yet been mentioned in the publications cited up to this point. We can also see from equation (9) that the failure of the equality of inertial and gravitational mass implied by Nordström's theory would reside in a second order effect in v/c. Nordström clearly believed it to be beyond recovery from then available experiments.

Finally we should note that Einstein and Nordström are using quite different versions of the requirement of the equality of inertial and gravitational mass. At this time, Einstein presumed that the total inertial mass would enter into the equality with the expectation that it would yield the independence of the vertical acceleration of a body in free fall from its horizontal velocity. We can only conjecture the precise sense Nordström had in mind, when he promised his theory would satisfy the equality, for he does not explain how the equality is expressed in his theory. My guess is that he took the rest mass to represent the body's inertial mass in the equality, for Nordström's equation (16) clearly shows that the motion of a massive particle in free fall is independent of its rest mass m. Under this reading Nordström's version of the equality entails the weaker observational requirement of the "uniqueness of free fall" defined above in Section 1.[29]

8. NORDSTRÖM'S FIRST THEORY ELABORATED

Nordström's first paper on his gravitation theory was followed fairly quickly by another (Nordström 1913a), submitted to *Annalen der Physik* in January 1913. This new paper largely ignored Einstein's objection although the paper bore the title "Inertial and gravitational mass in relativistic mechanics." The closing sections of this paper recapitulated the basic results of (Nordström 1912) with essentially notational differences only. In Section 6, Nordström's original field equation (12) was rewritten as

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}+\frac{\partial^2\Phi}{\partial u^2} = g\nu. \tag{12'}$$

As before, Φ was the gravitational potential. The rest density of matter was now represented by ν and Nordström explicitly named the new constant g the "gravitation

29 The equality of inertial and gravitational mass and the uniqueness of free fall are distinct from the principle of equivalence. Einstein's version of the principle has been routinely misrepresented since about 1920 in virtually all literatures. See (Norton 1985). It is stated correctly, however, in Nordström's addendum.

factor." The force equation was presented as a density in terms of the gravitational force per unit volume of matter $\mathfrak{K}_\mu^g$

$$\mathfrak{K}_\mu^g = -g\nu\frac{\partial\Phi}{\partial x_\mu}. \tag{18}$$

The only difference between this expression and the analogous one offered in the previous paper was the presence of the gravitation factor g. Since this factor was a constant and thus did not materially alter the physical content of either of the basic equations, Nordström might well have anticipated his readers' puzzlement over its use. He hastened to explain that, while g was a constant here, nothing ruled out the assumption that g might vary with the inner constitution of matter. The paper continued to derive the Φ dependence of rest mass m of equation (15). The new version of the relation now contained the gravitation factor g and read

$$m = m_0\exp\left(\frac{g\Phi}{c^2}\right).$$

Nordström no longer even mentioned the possibility of avoiding this dependence of m on Φ by positing the alternative force equation (17). The section continued with a brief treatment of the gravitational field stress-energy tensor and related quantities. It closed with a statement of the retarded potential solution of the field equation.

The final section 7 of the paper analyzed the motion of a point mass in free fall in an arbitrary static gravitational field. The analysis was qualified by repetition of his earlier observation that true point masses are impossible in his theory (Nordström 1912, 1129). In addition he noted that the particle's own field must be assumed to be vanishingly weak in relation to the external field. The bulk of the section is given over to a tedious but straightforward derivation of the analog of equation (9). Nordström considered a static field, that is one in which $\partial\Phi/\partial t = 0$, where t is the time coordinate of the coordinate system $(x, y, z, u = ict)$. He assumed the field homogeneous and acting only in the z-direction of the coordinate system. A point mass in free fall moves according to

$$\frac{d\mathfrak{v}_z}{dt} = -\left(1-\frac{\mathfrak{v}^2}{c^2}\right)g\frac{\partial\phi}{\partial z}, \quad \frac{d\mathfrak{v}_x}{dt} = 0, \quad \frac{d\mathfrak{v}_y}{dt} = 0, \tag{19}$$

where $\mathfrak{v}_x, \mathfrak{v}_y$ and $\mathfrak{v}_z$ are the components of the mass' velocity $\mathfrak{v}$.[30] At this point in the paper, readers of (Nordström 1912) might well suspect that the entire purpose of developing equation (19) was to enable statement of the objection of Einstein reported in that last paper's addendum. For, after observing that this result (19) tells

30 Notice that this result is more general than result (9), since it is not restricted to masses with vanishing vertical velocity, that is, masses whose motion satisfies the condition (7). Curiously Nordström's condition that the field be homogeneous, so that $\partial\Phi/\partial z = \text{constant}$, is invoked nowhere in the derivation or discussion of the result.

us that a body with horizontal velocity falls slower than one without, he concluded immediately that a rotating body must fall slower than a non-rotating body.

Because this example will be reappraised shortly, it is worth inserting the steps that Nordström must have assumed to arrive at this conclusion. In the simplest case, the axis of rotation of the body is aligned vertically in the static field. Each small element of the spinning body has a horizontal motion, due to the rotation. If each such element were independent, then the vertical acceleration of each would be given by the equation (19), so that each element would fall slower because of the horizontal velocity imparted by the rotation. If this result holds for each element, it seems unproblematic to conclude that it obtains for the whole, so that the vertical acceleration of fall of the body is diminished by its rotation.

While Nordström urged that this effect is much too small to be accessible to observation, he was more sanguine about the analogous effect on the acceleration of fall of a body by the independent motions of its molecules. Its possibility could not be denied, he said. However, in the penultimate paragraph of the paper, he anticipated that such an effect could be incorporated into his theory by allowing the gravitation factor g to depend on the molecular motion of the body. He pointed out that the rest energy of a body would also be influenced by this molecular motion.

The results Nordström recapitulated in Section 6 and 7 were not the major novelties of the paper. In fact the paper was intended to address a quite precise problem. The field equation (12') contained a density term ν. Nordström's problem was to identify what this term should be. The term—or, more precisely, $g\nu$—represented the gravitational field source density. According to Nordström's understanding of the equality of inertial and gravitational mass, ν must also represent the inertial properties of the source matter. The selection of such a term was not straightforward. For, drawing upon his own work and that of Laue and others, he knew that stressed bodies would exhibit inertial properties that were not reducible to the inertial properties of any individual masses that may compose them. Thus Nordström recognized that his gravitation theory must be developed by means of the theory of relativistic continua, in which stresses were treated. This had clearly been his program from the start. In a footnote to the first paragraph of his first paper on gravitation, Nordström (1912), at the mention of the equality inertial and gravitational mass, he foreshadowed his next paper

> By the equality of inertial and gravitational mass, I do not understand, however, that every inertial phenomenon is caused by an inertial and gravitational mass. For elastically stressed bodies, according to Laue ..., one recovers a quantity of motion [momentum] that cannot at all be reduced back to a mass. I will return to this question in a future communication.

The special behavior of stressed bodies proved to be of decisive importance for the development of Nordström's theory. Therefore, in the following section, I review the understanding of this behavior at the time of Nordström's work on gravitation. I will then return to (Nordström 1913a).

9. LAUE AND THE BEHAVIOR OF STRESSED BODIES

By 1911 it was apparent that a range of problems in the theory of relativity had a common core—they all involved the behavior of stressed bodies—and that a general theory of stressed bodies should be able to handle all of these problems in a unified format. The development of this general theory was largely the work of Laue and came from a synthesis and generalization of the work of many of his predecessors, including Einstein, Lorentz, Minkowski and Planck. The fullest expression of this general theory came in Laue (1911a) and was also incorporated into Laue (1911b), the first text book published on the new theory of relativity.[31] Three problems treated in Laue' s work give us a sense of the range of problems that Laue's work addressed.

9.1 Three Problems for Relativity Theory

In 1909, in a remarkably prescient paper, Lewis and Tolman (1909) set out to develop relativistic mechanics in a manner that was independent of electromagnetic theory using simple and vivid arguments. At this time relativity theory was almost invariably coupled with Lorentzian electrodynamics and its content was accessible essentially only to those with significant expertise in electrodynamics. Their exposition was marred, however, by an error in its closing pages (pp. 520–21). By this point, they had established the Lorentz transformation for forces transverse to the direction of motion. Specifically, if the force is f^0_{trans} in the rest frame, then the force f_{trans} measured in a frame moving at a fraction β of the speed of light is

$$f_{\text{trans}} = \sqrt{1-\beta^2} f^0_{\text{trans}}. \tag{20}$$

To recover the transformation formula for forces parallel to the direction of motion, the "longitudinal" direction, they considered the rigid, right angled lever of Figure 1. The arms ab and bc are of equal length and pivot about point b. In its rest frame two equal forces f act at points a and c, the first in direction bc, the second in direction ba. The level will not turn since there is no net turning couple about its pivot b. They then imagined the whole system in motion in the direction bc. They conclude—presumably directly from the principle of relativity—that the system must remain in equilibrium. Therefore the net turning couple about b must continue to vanish for the moving system, so that

$$\begin{pmatrix}\text{force}\\ \text{at } c\end{pmatrix}\begin{pmatrix}\text{length}\\ bc\end{pmatrix} + \begin{pmatrix}\text{force}\\ \text{at } a\end{pmatrix}\begin{pmatrix}\text{length}\\ ab\end{pmatrix} = 0.$$

Now, according to (20), the transverse force at c is diminished by the factor $\sqrt{1-\beta^2}$. The length of its lever arm bc is also contracted by the same factor,

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31 Presumably the two works were prepared together. (Laue 1911a) was submitted on 30 April, 1911. The introduction to (Laue 1911b) is dated May, 1911.

whereas the arm ab, being transverse to the motion, is uncontracted. Lewis and Tolman now concluded that equilibrium can only be maintained if the longitudinal force f_{long} at a transforms according to

$$f_{\text{long}} = (1-\beta^2) f^0_{\text{long}} . \quad (21)$$

This conclusion comes from an argument so simple that one would hardly suspect it. What they did not point out, however, was that its conclusion (21) contradicted the then standard expositions of relativity theory (e.g. Einstein 1907a, 448) according to which (20) is correct but (21) should be replaced by

$$f_{\text{long}} = f^0_{\text{long}} . \quad (22)$$

We now see the problem in its starkest form. If we apply the standard transformation formulae (20) and (22) to the case of Lewis and Tolman's bent lever we seem driven to a curious conclusion. We have a system at equilibrium in its rest frame which now forfeits that equilibrium in a moving frame through the appearance of a non-vanishing turning couple. Indeed we seem to have a violation of the principle of relativity, for the presence of this turning couple should yield an experimental indication of the motion of the system.

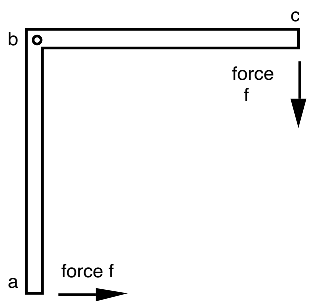

Figure 1: Lewis and Tolman's Bent Lever

The second problem is, at first glance, quite unrelated to the Lewis and Tolman bent lever. Under a classical analysis, one expects that a charged, parallel plate condenser can experience a net turning couple if it is set in motion through the aether. In a classic experiment, Trouton and Noble (1903) sought to detect the turning couple acting on a charged condenser due to its motion with the earth. Their null result is celebrated. Just as in the case of the Lewis and Tolman bent lever, the problem is to see how relativity theory allows one to predict this null result, which otherwise would contradict the principle of relativity. In fact, as Laue (1911c, 517) and others soon pointed out, the two problems were closely connected. In its rest frame, the Trouton-Noble condenser was simply a rigid system of two parallel plates with an electric force acting on each plate in such a way that the entire system was in equilibrium. If that equilibrium system was set into motion, under either a classical or relativistic analysis, the electric forces would transform according to the Lorentz transformation (20) and (22). Unless the direction of motion imparted was parallel or exactly perpendicular to the plates, the net effect would be exactly the same as the Lewis and Tolman bent lever. A non-vanishing turning couple is predicted which deprives the system of equilibrium. The couple ought to be detectable in violation of the principle of relativity.[32]

The third problem concerns the theory of electrons. The decade preceding 1911 had seen considerable work on the problem of providing a model for the electron. Best known of these were the models of Lorentz and Abraham, which depicted electrons as electrically charged spheres with varying properties. The general problem was to show that the relativistic dynamics of an acceptable model of the electron would coincide with the relativistic dynamics of a point mass. There were a range of difficulties to be addressed here. In introducing his (Laue 1911a, 524–25), Laue recalled a brief exchange between Ehrenfest and Einstein. In a short note, Ehrenfest (1907) had drawn on work of Abraham that raised the possibility of troubling behavior by an electron of non-spherical or non-ellipsoidal shape when at rest. It was suggested that such an electron cannot persist in uniform translational motion unless forces are applied to it.[33] We might note that such a result would violate not only the principle of inertia in the dynamics of point masses but also the principle of relativity. Einstein's reply (1907b) was more a promise than resolution, although he ultimately proved correct. He pointed out that Ehrenfest's model of the electron was incomplete. One must also posit that the electron's charge was carried by a rigid frame, stressed to counteract the forces of self repulsion of the charge distribution. Ehrenfest's problem could not be solved until a theory of such frames was developed.

Finally another aspect of the problem of the relativistic dynamics of electrons was the notorious question of electromagnetic mass. If one computed the total momentum and energy of the electromagnetic field of an electron, the result universally accepted at this time was the one reported in (Laue 1911b, 98):

$$\begin{pmatrix}\text{Total field}\\ \text{momentum}\end{pmatrix} = \frac{4}{3}\frac{1}{c^2}\begin{pmatrix}\text{Total field}\\ \text{energy}\end{pmatrix}\begin{pmatrix}\text{electron}\\ \text{velocity}\end{pmatrix}. \tag{23}$$

The conflict with the relativistic dynamics of point masses arose if one now posited that all the energy and momentum of the electron resides in its electromagnetic field. For one must then identify $\frac{1}{c^2}\begin{pmatrix}\text{Total field}\\ \text{energy}\end{pmatrix}$, the electromagnetic mass of the electron,

32 For further extensive discussion of the Trouton-Noble experiment and its aether theoretic treatment by Lorentz, see (Janssen 1995).

33 In a footnote, Ehrenfest pointed out the analogy to the turning couple induced on a charged condenser and reviewed the then current explanation of it absence in terms of molecular forces.

as the total inertial mass of the electron, so that equation (23) tells us that the momentum of an electron is $4/3$ the product of its mass and velocity. The canonical resolution of this difficulty, as stated for example in (Pauli 1921, 185–86), is that such a purely electromagnetic account of the dynamics of the electron is inadmissible. As Einstein (1907b) urged, there must be also stresses of a non-electromagnetic character within the electron.[34] The puzzle Laue addressed in 1911 was to find very general circumstances under which the dynamics of such an electron would agree with the relativistic dynamics of point masses.

9.2 The General-Stress Energy Tensor

The focus of Laue's treatment of stressed bodies in his (1911a) and (1911b) lay in a general stress-energy tensor.[35] While Minkowski (1908, § 13) had introduced the four dimensional stress-energy tensor at the birth of four dimensional methods in relativity theory, his use of the tensor was restricted to the special case of the electromagnetic field. Laue's 1911 work concentrated on extending the use of this tensor to the most general domain. The properties of this tensor and its behavior under Lorentz transformation summarized a great deal of the then current knowledge of the behavior of stressed bodies. Laue (1911a) uses a coordinate system $(x, y, z, l = ict)$ so that the components of the stress energy tensor $T_{\mu\nu}$ have the following interpretations:

$$\begin{pmatrix} T_{xx} = p_{xx} & T_{xy} = p_{xy} & T_{xz} = p_{xz} & T_{xl} = -ic\mathfrak{g}_x \\ T_{yx} = p_{yx} & T_{yy} = p_{yy} & T_{yz} = p_{yz} & T_{yl} = -ic\mathfrak{g}_y \\ T_{zx} = p_{zx} & T_{zy} = p_{zy} & T_{zz} = p_{zz} & T_{zl} = -ic\mathfrak{g}_z \\ T_{lx} = \frac{i}{c}\mathfrak{S}_x & T_{ly} = \frac{i}{c}\mathfrak{S}_y & T_{lz} = \frac{i}{c}\mathfrak{S}_z & T_{ll} = -W \end{pmatrix}.$$

The three dimensional tensor $p_{ik}(i, k = 1, 2, 3)$ is the familiar stress tensor. The vector $\mathfrak{g} = (\mathfrak{g}_x, \mathfrak{g}_y, \mathfrak{g}_z)$ represents the momentum density. The vector $\mathfrak{S} = (\mathfrak{S}_x, \mathfrak{S}_y, \mathfrak{S}_z)$ is the energy flux. W is the energy density.

The most fundamental result of relativistic dynamics is Einstein's celebrated inertia of energy according to which every quantity of energy E is associated with an inertial mass (E/c^2). The symmetry of Laue's tensor entails a result closely con-

34 While these stresses are needed to preserve the mechanical equilibrium of the electron, Rohrlich (1960) showed that they were not needed to eliminate the extraneous factor of 4/3 in equation (23). He showed that the standard derivation of (23) was erroneous and that the correct derivation did not yield the troubling factor of 4/3.

35 The label "stress-energy tensor" is anachronistic. Laue had no special name for the tensor other than the generic "world tensor," which, according to the text book exposition of Laue (1911b, § 13) described any structure which transformed as what we would now call a second rank, symmetric tensor. Notice that the term "tensor" was still restricted at this time to what we would now call second rank tensors and even then usually to symmetric, second rank tensors. See (Norton 1992, Appendix).

nected with Einstein's inertia of energy and attributed to Planck by Laue (1911a, 530). We have $(T_{xl}, T_{yl}, T_{zl}) = (T_{lx}, T_{ly}, T_{lz})$ which immediately leads to

$$g = \frac{1}{c^2}\mathfrak{S} \tag{24}$$

This tells us that whenever there is an energy flux $\mathfrak{S}$ present in a body then there is an associated momentum density g.

As emphasized in (Laue 1911c), this result is already sufficient to resolve the first of the three problems described above in Section 9.1, the Lewis and Tolman bent lever. Notice first that Figure 1 does not display all the forces present. There must be reaction forces present at the pivot point b to preserve equilibrium in the rest frame. See Figure 2, which also includes the effect of the motion of the system at velocity v in the direction bc. When the lever moves in the direction bc, then work is done by the force f at point a which acts in the direction bc. The energy of this work is transmitted along the arm ab as an energy current of magnitude fv and is lost at the pivot point b as work done against the reaction force that acts in the direction cb. This energy current fv in the arm ab must be associated with a momentum fv/c^2, according to (24) when integrated over the volume of the arm ab, and this momentum will be directed from a towards b. As Laue (1911c) showed, a short calculation reveals that this momentum provides precisely the additional turning couple needed to return the moving system to equilibrium. Notice that the force at c and its associated reaction force at b are directed transverse to the motion so no work is done by them.[36]

The essential and entirely non-classical part of the analysis resides in the result that the there is an additional momentum present in the moving arm ab because it is under the influence of a shear stress due to the force f at a and the corresponding reaction force at b. As Laue (1911c, 517) and Pauli (1921, 128–29) point out, exactly this same relativistic effect explains the absence of net turning couple in the Trouton-Noble condenser. The condenser's dielectric must be stressed in reaction to the attractive forces between the oppositely charged plates. The additional momentum associated with these non-electromagnetic stresses provides the additional turning couple required to preserve the equilibrium of the moving condenser.

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36 Assume that the level arms are of unit length at rest so that the arm bc contracts to length $\sqrt{1 - v^2/c^2}$ when the system moves at velocity v in direction bc. The turning couple about a point b' at rest and instantaneously coincident with the moving pivot point b, due to the applied forces alone is

$$f(1 - v^2/c^2) - f = -f(v^2/c^2)$$

where a positive couple is in the clockwise direction. The relativistic momentum fv/c^2 generates angular momentum about the point b'. Since the distance of the arm ab from b' is growing at the rate of v, this angular momentum is increasing at a rate $v(fv/c^2) = f(v^2/c^2)$, which is exactly the turning couple needed to balance the couple due to the applied forces.

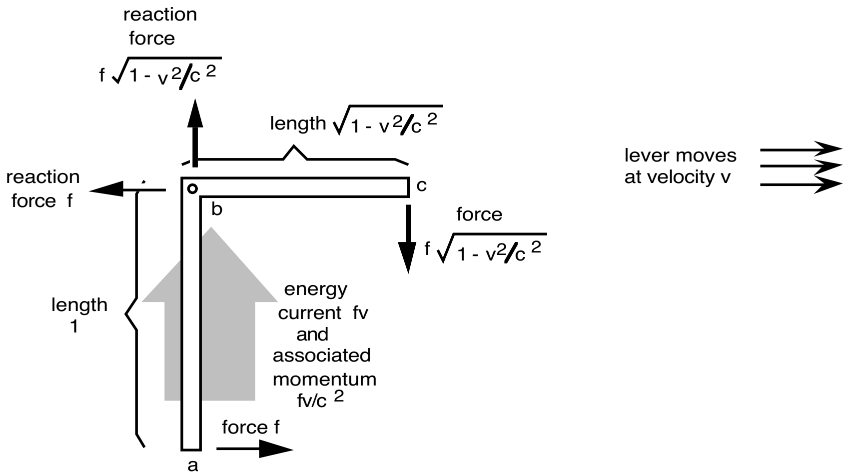

Figure 2: Lewis and Tolman's Bent Lever Showing Reaction Forces and Effects of Motion

While this analysis satisfactorily resolves at least the case of the Lewis and Tolman lever, more complicated cases will require a clearer statement of the relationship between the stresses in a moving body and the momentum associated with it. These results were derived directly by Laue (1911a, 531–32) from the Lorentz transformation of the components of the stress-energy tensor. In the rest frame of the stressed matter distribution, the matter has energy density W^0 and a stress tensor p^0_{ik}, for $i, k = 1, 2, 3$. The momentum density $\mathfrak{g}$ and energy flux $\mathfrak{S}$ vanish. Transforming to a frame of reference moving at velocity q in the x direction, from a direct application of the Lorentz transformation formula for a tensor, Laue recovered results that included

$$\mathfrak{g}_x = \frac{q}{c^2 - q^2}(p^0_{xx} + W^0), \quad \mathfrak{g}_y = \frac{q}{c\sqrt{c^2 - q^2}}p^0_{xy}, \quad W = \frac{c^2 W^0 + q^2 p^0_{xx}}{c^2 - q^2}. \qquad (25)$$

The first two equations show that, in a moving body, there is a momentum density associated with both normal stresses (p^0_{xx}) and with shear stresses (p^0_{xy}). The Lewis and Tolman lever is a case in which a momentum density is associated with shear stresses in a moving body in accord with (25). The third equation shows that there is an energy density associated with normal stresses in moving body.

This last result, in a form integrated over a whole stressed body, had already been investigated and clearly stated by Einstein (1907c, § 1), as a part of his continuing analysis of the inertia of energy. He gave the result a very plausible, intuitive basis, relating it directly to the relativity of simultaneity. He imagined a rigid body in uniform translational motion. At some single instant in the body's rest frame, an equilibrium state of stress appears in the body. Since it appears at a single instant, the new forces do not alter the state of motion of the body. However in the frame in which the body moves, because of the relativity of simultaneity, these new forces do not appear simultaneously over the entire body. Thus there is a brief period of disequilibrium of forces during which net work is done on the body. This new work is exactly the energy associated with the stresses p^0_{xx} in the equation (25). Einstein (1907a, § 2) continued with an example similar in structure to the Trouton-Noble condenser—a rigid body, in uniform motion, carrying an electric charge distribution. The forces between the charges carried stress the rigid body, so that there is an energy associated with these stresses. Einstein showed that this latter energy was essential. Otherwise the energy of the moving body would depend on the direction of its motion which would lead to a contradiction.[37]

Laue (1911, § 2) continued his treatment of the transformation formulae (25) by restating them for an extended body. In particular, integration of (25) over such a body revealed relationships between the rest energy E^0 of the body and its energy E and momentum $\mathfrak{G}$ in the frame of reference in which the body moves at velocity q. Writing the three components of the body's ordinary velocity as (q_1, q_2, q_3), he recovered[38]

$$E = \frac{c}{\sqrt{c^2-q^2}}\left\{E^0 + \frac{1}{c^2}q_i q_k \int p^0_{ik} dV^0\right\}$$

$$\mathfrak{G}_i = \frac{q_i}{c\sqrt{c^2-q^2}}\left\{E^0 + \frac{1}{q^2}q_j q_k \int p^0_{jk} dV^0\right\} + \frac{1}{c^2}\left\{q_k \int p^0_{ik} dV^0 - \frac{q_i}{q^2} q_j q_k \int p^0_{jk} dV^0\right\}. \tag{26}$$

The expression for momentum had an immediate and important consequence. In general, whenever the body was stressed so that the stress tensor p^0_{ik} does not vanish, the momentum $\mathfrak{G}_i$ of the body will not be in the same direction as its velocity q_i. This was exemplified in the Lewis and Tolman lever. Although it was set in motion in the direction bc, the presence of stresses in the arm ab led to a momentum in that arm

37 Specifically, in the body's rest frame, the body can rotate infinitely slowly without application of any forces. By the principle of relativity, this same motion will be possible if the body is in uniform translational motion as well. However in this latter case the kinetic energy of the body would alter according to its orientation as it rotates. Since no forces were applied, this would violate the "energy principle," the law of conservation of energy. Notice that the rotation is infinitely slow, so that it does not contribute to the body's kinetic energy.

38 V^0 is the rest volume of the body and $i, j, k = 1, 2, 3$. I have simplified Laue's opaque notation by introducing an index notation, where Laue used round and square brackets to represent various products. For example, where I would write $q_i q_k p^0_{ik}$, he would write "$(q[qp^0])$."

directed transverse to the motion. If this momentum is added vectorially to the momentum of the inertial mass of the lever, the resultant total momentum vector will not be parallel to the direction of motion.

Laue was now in a position to restate the analysis given for the Lewis and Tolman lever in a way that would apply to general systems. This was the principle of burden of (Laue 1911a, § 3 and § 4). To begin, Laue introduced a new three dimensional stress tensor. In a body at rest, the time rate of change of momentum density $\partial \mathfrak{g}_i / \partial t$ is given by the negative divergence of the tensor p_{ik}:

$$\frac{\partial \mathfrak{g}_i}{\partial t} = -\frac{\partial p_{ik}}{\partial x_k}.$$

However if one wishes to investigate the time rate of change of momentum density in a moving body, one must replace the partial time derivative $\frac{\partial}{\partial t}$ with a total time derivative coordinated to the motion, $\frac{d}{dt} = \frac{\partial}{\partial t} + q_x \frac{\partial}{\partial x} + q_y \frac{\partial}{\partial y} + q_z \frac{\partial}{\partial z}$. Laue was able to show that the relevant time rate of change of momentum was given as the negative divergence of a new tensor t_{ik}

$$\frac{d\mathfrak{g}_i}{dt} = -\frac{\partial t_{ik}}{\partial x_k}.$$

where this "tensor of elastic stresses" was defined by

$$t_{ik} = p_{ik} + \mathfrak{g}_i q_k.$$

Note in particular that t_{ik} will not in general be symmetric since the momentum density $\mathfrak{g}_i$ will not in general be parallel to the velocity q_i.

The lack of symmetry of t_{ik} is a cause of momentary concern, for it is exactly the symmetry of p_{ik} that enables recovery, in effect, of the law of conservation of angular momentum. More precisely, the symmetry of the stress tensor is needed for the standard derivation of the result that the time rate of change of angular momentum of a body is equal to the total turning couple impressed on its surface. Laue proceeds to show, however, that this asymmetry does not threaten recovery of this law and is, in fact essential for it.[39] He writes the time rate of change of total angular momentum $\mathfrak{L}_i$ of a moving body as[40]

39 Laue calls § 4, which contains this discussion, "the area law." I presume this is a reference to Kepler's second law of planetary motion, which amount to a statement of the conservation of angular momentum for planetary motion.

40 dV is a volume element of the body. I make no apology at this point for shielding the reader from Laue's notation, which has become more than opaque. Laue now uses square brackets to represent vector products, where earlier they represented an inner product of vector and tensor. ε_{ikl} is the fully antisymmetric Levi-Civita tensor, so that $\varepsilon_{ikl} A_k B_l$ is the vector product of two vectors A_k and B_l.

$$\frac{d\mathfrak{L}_i}{dt} = \int \varepsilon_{ikl}\mathfrak{r}_k\frac{d\mathfrak{g}_l}{dt} + \varepsilon_{ikl}q_k\mathfrak{g}_l dV = \int -\varepsilon_{ikl}\mathfrak{r}_k\frac{\partial t_{lm}}{\partial x_m} + \varepsilon_{ikl}q_k\mathfrak{g}_l dV. \tag{27}$$

Were the tensor t_{ik} symmetric, then the integration of the first term alone could be carried out using a version of Gauss' theorem. One could then arrive at the result that the total rate of change of momentum of the body is given by the turning couple applied to its surface. Thus if there is no applied couple, angular momentum would be conserved. However the tensor t_{ik} is not symmetric, and so an integral over the first term leaves a residual rate of change of angular momentum even when no turning couple is applied to the body. Fortunately there is a second term in the integrals of (27) that results from allowing for the use of the total time derivative. It is the vector product of the velocity q_i and momentum density $\mathfrak{g}_i$. This term would not be present in a classical analysis since these two vectors would then be parallel so that their vector product would vanish. In the relativistic context, this is not the case. This term corresponds exactly to the stress induced momentum in the arm ab of the Lewis and Tolman lever. This extra term exactly cancels the residual rate of change of angular momentum of the first term, restoring the desired result, the rate of change of angular momentum equals the externally applied turning couple.

9.3 Laue's "Complete Static Systems"

The last of the group of results developed in (Laue 1911a, § 5) proved to be the most important for the longer term development of Nordström's theory of gravitation. Laue had shown clearly just how different the behavior of stressed and unstressed bodies in relativity theory could be. He now sought to delineate circumstances in which the presence of stresses within a body would not affect its overall dynamics. Such was the case of a "complete static system," which Laue defined as follows:

> We understand by this term such a system which is in static equilibrium in any justified reference system K^0, without sustaining an interaction with other bodies.

This definition is somewhat elusive and the corresponding definition in (Laue 1911b, 168–69) is similar but even briefer. In both cases, however, Laue immediately gave the same example of such a system, "an electrostatic field including all its charge carriers." This example and the definition leaves open the question of whether a body spinning at constant speed and not interacting with any other bodies is a complete static system. Such a body is in equilibrium and static in the sense that its properties are not changing with time, especially if the body spins around an axis of rotational symmetry. Tolman (1934, 81) gave a clearer definition:

> And in general we shall understand by a complete static system, an entire structure which can remain in a permanent state of rest with respect to a set of proper coordinates S^0 without the necessity for any forces from the outside.

He clearly understood this definition to rule out rotating bodies, for he noted a few lines later "the velocity of all parts of the system is zero in these coordinates $[S^0]$".

Tolman used this to justify the condition that the momentum density in the rest frame vanishes at every point

$$\mathfrak{g}^0 = 0. \tag{28}$$

Presumably Laue agreed for he also invoked this condition. From it, both Laue and Tolman derived the fundamental result characteristic of complete static systems:

$$\int p_{ik}^0 dV^0 = 0, \tag{29}$$

where the integral extends over the rest volume V^0 of the whole body. Laue allowed, in effect, that his conception of a complete static system could be relaxed without compromising the recovery of (29). For in a footnote (Laue 1911a, 540) to the example of an electrostatic field with its charge carriers, he noted that one could also consider the case of electrostatic-magnetostatic fields. Even though (28) failed to obtain for this case, the time derivative of $\mathfrak{g}^0$ did vanish which still allowed the derivation of (29).

This fundamental property (29) of complete static systems greatly simplified the expression (26) for the energy and momentum of a stressed body. Through (29) all the terms explicitly dependent on stresses vanish so that

$$\begin{aligned} E &= \frac{c}{\sqrt{c^2-q^2}}E^0, \\ \mathfrak{G}_i &= \frac{q_i}{c\sqrt{c^2-q^2}}E^0. \end{aligned} \tag{30}$$

As Laue pointed out, these expressions coincide precisely with those of a point mass with rest mass $m^0 = (E^0/c^2)$. Moreover under quasi-stationary acceleration—that is acceleration in which "the inner state $(E^0, \mathbf{p}^0)$ is not noticeably changed"—a complete static system will behave exactly like a point mass.

Laue could now offer a full resolution of the remaining problems described above in Section 9.1. An electron together with its field is a complete static system, he noted, no matter how it may be formed. As a result it will behave like a point mass, as long as its acceleration is quasi-stationary. In particular it will sustain inertial motion without the need for impressed forces. While Laue did not explicitly mention the problem of relating the electron's total field momentum to its inertial mass, Laue's result (30) resolves whatever difficulty might arise for the overall behavior of an electron. For however the electron may be constructed, as long as it forms a complete static system, equation (30) shows that the extraneous factor of 4/3 in equation (23) cannot appear. Finally, the Trouton-Noble condenser is a complete static system. While neither the momentum of its electromagnetic field or of its stressed mechanical structure will lie in the direction of its motion, equation (30) shows that the combined momentum $\mathfrak{G}_i$ will lie parallel to the velocity q_i, so that there is no net turning couple acting on the condenser.[41]

10. THE DEFINITION OF INERTIAL MASS IN NORDSTRÖM'S FIRST THEORY

What had emerged clearly from Laue's work was that the inertial properties of bodies could not be explained solely in terms of their rest masses and velocities, if the bodies were stressed. For Laue's equation (26) showed that the momentum of a moving body would be changed merely by the imposition of a stress, even though that stress need not deform the body or perform net work on it. Nordström clearly had results such as these in mind when he laid out the project of his (Nordström 1913a, 856–57). Laue and Herglotz, he reported, had constructed the entire mechanics of extended bodies without exploiting the concept of inertial mass. That concept, he continued, was neither necessary nor sufficient to represent the inertial properties of stressed matter. This now seems to overstate the difficulty, for Laue's entire system depended upon Einstein's result of the inertia of energy. Nonetheless nowhere did Laue's mechanics of stressed bodies provide a single quantity that represented *the* inertial mass of a stressed body.

It was to this last omission that Nordström planned to direct his paper. It was important, he urged, to develop a notion of the inertial mass of matter for the development of a gravitation theory. Such a theory must be based on the "unity of essence"[42] of inertia and gravity. He promised to treat the relativistic mechanics of deformable bodies in such a way that it would reveal a concept of inertial mass suitable for use in a theory of gravitation.

Nordström's analysis was embedded in a lengthy treatment of the mechanics of deformable bodies whose details will not be recapitulated here. Its basic supposition, however, was that the stress energy tensor $T_{\mu\nu}$ of a body with an arbitrary state of motion and stress would be given as the sum of two symmetric tensors (p. 858)

$$T_{\mu\nu} = p_{\mu\nu} + \nu\mathfrak{B}_{\mu}\mathfrak{B}_{\nu}. \tag{31}$$

The second tensor, $\nu\mathfrak{B}_{\mu}\mathfrak{B}_{\nu}$, he called the "material tensor." It represented the contribution to the total stress tensor from a matter distribution with rest mass density ν and four velocity $\mathfrak{B}_{\mu}$. The first tensor, $p_{\mu\nu}$, he called the "elastic stress tensor." It represented the stresses in the matter distribution. In the rest frame of the matter distribution, Nordström wrote the elastic stress tensor as (p. 863)

41 However this result did not end Laue's analysis of the Trouton-Noble experiment. See (Laue 1912).

42 *Wesenseinheit*. The term is sufficiently strong and idiosyncratic for it to be noteworthy that, so far as I know, Einstein was the only other figure from this period who used even a related term in connection with inertia and gravitation. In a paper cited earlier in (Nordström 1912, 1126), Einstein (1912d, 1063) had talked of the "equality of essence" (*Wesensgleichheit*) of inertial and gravitational mass. Einstein used the term again twice in later discussion. See (Norton 1985, 233).

$$\begin{bmatrix} p^0_{xx} & p^0_{xy} & p^0_{xz} & 0 \\ p^0_{yx} & p^0_{yy} & p^0_{yz} & 0 \\ p^0_{zx} & p^0_{zy} & p^0_{zz} & 0 \\ 0 & 0 & 0 & p^0_{uu}. \end{bmatrix}$$

The six zero-valued components in this matrix represent the momentum density and energy current due to the presence of stresses. They must vanish, Nordström pointed out, since stresses cannot be responsible for a momentum or energy current in the rest frame.[43] In particular, Nordström identified the component p^0_{uu} as a Lorentz invariant.[44]

In the crucial Section 4, "Definition of Inertial Mass," Nordström turned his attention to the (uu) component of equation (31) in the rest frame. This equation gave an expression for the Lorentz invariant rest energy density Ψ in terms of the sum of two invariant quantities[45]

$$\Psi = -p^0_{uu} + c^2\nu. \tag{32}$$

This equation gave simplest expression to the quantity of fundamental interest to Nordström's whole paper, the density ν, which would provide the source for the gravitational field equation. This density would be determined once Ψ and p^0_{uu} were fixed. However, while the rest energy density Ψ was a "defined quantity," it was not so clear how p^0_{uu} was to be determined. It represented an energy density associated with the stresses. Clearly if there were no stresses in the material, then this energy would have to be zero. But what if there were stresses?

To proceed Nordström considered a special case, a body in which there is an isotropic, normal pressure. In this case, Nordström continued, it is possible to fix the value of p^0_{uu} in such a way that the density ν can be determined. The elastic stress tensor could be generated out of a single scalar invariant, which I will write here as p, so that the elastic stress tensor in the rest frame is given by

43 In Section 5, Nordström augmented his analysis by considering the effect of heat conduction. This was represented by a third symmetric tensor, $w_{\mu\nu}$, whose *only* non-zero components in the rest frame were exactly these six components. Thus heat conduction was represented by an energy current and associated momentum density which did not arise from stresses and which had no associated energy density in the rest frame.

44 This followed easily from the fact that the tensor $p_{\mu\nu}$ twice contracted with the four velocity $\mathfrak{B}_\mu$ yields a Lorentz invariant, $p_{\mu\nu}\mathfrak{B}_\mu\mathfrak{B}_\nu$, which can be evaluated in the rest frame, where $\mathfrak{B}_\mu = (0, 0, 0, ic)$, and turns out to be $-c^2 p^0_{uu}$.

45 The presence of the negative sign follows from the use of a coordinate system in which the fourth coordinate is $x_4 = u = ict$. Therefore $T_{44} = T_{uu} = -$ (energy density).

$$\begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \tag{33}$$

With this particular choice of stress tensor, Nordström pointed out, there is no momentum density associated with the stresses when the body is in motion. We can confirm this conclusion merely by inspecting the matrix (33). Since p is an invariant, the matrix will transform back into itself under Lorentz transformation. Therefore in all frames of reference, the six components (p_{14}, p_{24}, p_{34}) and (p_{41}, p_{42}, p_{43}) will remain zero. But these six components between them represent the momentum density and energy current due the stresses. Thus any momentum density present in the body will be due to the density ν.[46]

At this point, the reader might expect Nordström to recommend that one set p^0_{uu} in the general case in such a way that there are no momentum densities associated with stresses. Nordström informs us, however, that he could find no natural way of doing this. As a result, he urged that the "simplest and most expedient definition" lies in setting

$$c^2\nu = \Psi \tag{34}$$

so that $p^0_{uu} = 0$. To quell any concern that this choice had been made with undue haste, Nordström continued by asserting that the factual content of relativistic mechanics is unaffected by the choice of quantity that represents inertial mass. It is only when weight is assigned to inertial masses, such as in his gravitation theory, that the choice becomes important.

The reader who has followed the development of Nordström's argument up to this point cannot fail to be perplexed at the indirectness of what is the core of the entire paper! There are three problems. First, the choice of ν as given in (34) seems unchallengeable as the correct expression for the rest density of inertial mass. It merely sets this density equal to $1/c^2$ times the rest energy density—exactly as one would expect from Einstein's celebrated result of the inertia of energy. Indeed, any other division of total rest energy Ψ between the two terms of (32) would force us to say that ν does not represent the total inertial rest mass density, for there would be another part of the body's energy it does not embrace. Second, no argument is given for the claim that the choice of ν has no effect on the factual content of relativistic mechanics.[47] Finally, even if this second point is correct, it hardly seems worth much attention since the choice of expression for ν does significantly affect the factual content of gravitation theory.

The clue that explains the vagaries of Nordström's analysis lies in his citation of his own (Nordström 1911). There one finds an elaborate analysis of the relativistic

46 Unless, of course, heat conduction is present.

mechanics of the special case of a body with isotropic normal stresses—exactly the special case considered above. The analysis began by representing the stresses through a tensor of form (33). Nordström then showed the effect of arbitrarily resetting the value of p^0_{uu}. Reverting to the notation of (Nordström 1911), Nordström imagined that the p^0_{uu} term of (33) is replaced by some arbitrary U_0. He showed that the effect of this substitution is simply to replace the rest mass density γ (the analog of v in the 1913 paper) in the equations of the theory with an augmented

$$\gamma' = \gamma + \frac{p + U_0}{c^2}$$

without otherwise altering the theory's relations. He was able to conclude that setting $U_0 = p$ "is not a specialization of the theory, but only a specialization of concepts." In the introduction to (Nordström 1911), he had announced his plan to extend this analysis to the more general case with tangential stresses in another paper. Presumably the discussion of Section 4 in (Nordström 1913a) was intended to inform his readers that he was now unable to make good on his earlier plan. Indeed the remarks that seemed puzzling are merely a synopsis of some of the major points of (Nordström 1911). That the choice of (34) does not affect the factual content of relativistic mechanics is merely an extension of the result developed in detail in (Nordström 1911). It had become something of a moot point, however, in the context of gravitation theory.

To sum up, Nordström's choice of source density v was given by equation (34) and it was this result that gave meaning to the quantity v in the final sections of the paper in which his gravitation theory was recapitulated. We can give this quantity more transparent form by writing it in a manifestly covariant manner[48]

$$\nu = -\frac{1}{c^2} T_{\mu\nu} \mathfrak{B}_\mu \mathfrak{B}_\nu. \tag{35}$$

Natural as this choice seemed to Nordström, it was Einstein who shortly proclaimed that another term derived from the stress energy tensor was the only viable candidate and that this unique candidate led to disastrous results.

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47 On reflection, however, I think the result not surprising. Barring special routes such as might be provided through gravitation theory, we have no independent access to the energy represented by the term p^0_{uu}. For example, in so far as this energy is able to generate inertial effects, such as through generation of a momentum density, it is only through its contribution to the sum $T_{uu} = -p^0_{uu} + c^2\nu$. The momentum density follows from the Lorentz transformation of the tensor $T_{\mu\nu}$. How we envisage the energy divided between the two terms of this sum will be immaterial to the final density yielded.

48 While $-c^2\nu$ is the trace of the material tensor $\nu\mathfrak{B}_\mu\mathfrak{B}_\nu$, this quantity $-c^2\nu$ is not the trace of the full tensor $T_{\mu\nu}$ as given in (31). This latter trace would contain terms in p^0_{xx}, etc.

11. EINSTEIN OBJECTS AGAIN

By early 1913, Einstein's work on his own gravitation theory had taken a dramatic turn. With his return to Zurich in August 1912, he had begun a collaboration with his old friend Marcel Grossmann. It culminated in the first sketch of his general theory of relativity, (Einstein and Grossmann 1913), the so-called "*Entwurf*" paper. This work furnished his colleagues all the essential elements of the completed theory of 1915, excepting generally covariant gravitational field equations.[49] While we now know that this work would soon be Einstein' s most celebrated achievement, the Einstein of 1913 could not count on such a jubilant reception for his new theory. He had already survived a bitter dispute with Abraham over the variability of the speed of light in his earlier theory of gravitation. And Einstein sensed that the lack of general covariance of his gravitational field equations was a serious defect of the theory which would attract justifiable criticism.

There was one aspect of the theory which dogged it for many years, its very great complexity compared with other gravitation theories. In particular, in representing gravitation by a metric tensor, Einstein had, in effect, decided to replace the single scalar potential of gravitation theories such as Newton's and Nordström's, with ten gravitational potentials, the components of the metric tensor. This concern was addressed squarely by Einstein in Section 7 of his part of the *Entwurf* paper. It was entitled "Can the gravitational field be reduced to a scalar?" Einstein believed he could answer this question decisively in the negative, thereby, of course, ruling out not just Nordström's theory of gravitation, but any relativistic gravitation theory which represented the gravitational field by a scalar potential.

Einstein's analysis revealed that he agreed with Nordström's assessment of the importance of Laue's work for gravitation theory. However he felt that Laue's work, in conjunction with the requirement of the equality of inertial and gravitational mass, pointed unambiguously to a different quantity as the gravitational source density. That was the trace of the stress-energy tensor. He proposed that a scalar theory would be based on the equation of motion for a point mass

$$\delta\left\{\int \Phi ds\right\} = 0 \tag{36}$$

where Φ is the gravitational potential, ds is the spacetime line element of special relativity and δ represents a variation of the mass' world line. He continued, tacitly comparing the scalar theory with his new *Entwurf* theory:

> Here also material processes of arbitrary kind are characterized by a stress-energy tensor $T_{\mu\nu}$. However in this approach a scalar determines the interaction between the gravitational field and material processes. This scalar, as Herr Laue has made me aware, can only be

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49 For an account of this episode, see (Norton 1984).

$$\sum_{\mu} T_{\mu\mu} = P$$

I want to call it "Laue's scalar." Then one can do justice to the law of the equivalence of inertial and gravitational mass here also up to a certain degree. That is, Herr Laue has pointed my attention to the fact that, for a closed system,

$$\int P dV = \int T_{44} d\tau.$$

From this one sees that the weight of a closed system is determined by its total energy according to this approach as well.

Recall that Einstein's version of the requirement of the equality of inertial and gravitational mass seeks to use the total energy of a system as a measure of it as a gravitational source. The selection of P, the trace of the stress energy tensor, does this for the special case of one of Laue's complete static systems. For such a system, the integral of the trace P over the spatial volume V of the system is equal to the negative value of the total energy of the system[50] $\int T_{44} dV$ since

$$\int P dV = \int T_{11} + T_{22} + T_{33} + T_{44} dV = \int T_{44} dV. \tag{37}$$

The three terms in T_{11}, T_{22} and T_{33} in the integral vanish because of the fundamental property (29) of complete static systems. Notice that Einstein can only say he does justice to the equality of inertial and gravitational mass "up to a certain degree," since this result is known to hold only for complete static systems and then only in their rest frames.

Einstein's wording indicates direct personal communication from Laue. concerning the stress-energy tensor and complete static systems. Such personal communication is entirely compatible with the fact that both Einstein and Laue were then in Zurich, with Einstein at the ETH and Laue at the University of Zurich. Below, in Section 15, I will argue that there is evidence that, prior to his move to Zurich, Einstein was unaware of the particular application of Laue's work discussed here by him.

Einstein continued his analysis by arguing that this choice of gravitational source density was disastrous. It leads to a violation of the law of conservation of energy. He wrote:

The weight of a system that is not closed would depend however on the orthogonal stresses T_{11} etc. to which the system is subjected. From this there arise consequences which seem to me unacceptable as will be shown in the example of cavity radiation.[51]

For radiation in a vacuum it is well known that the scalar P vanishes. If the radiation is enclosed in a massless, mirrored box, then its walls experience tensile stresses that cause

50 Presumably Einstein's "$d\tau$" is a misprint and should read dV. In a coordinate system in which $x_4 = ict$, $T_{44} = -$(energy density), so that $\int T_{44} dV$ is the negative value of the total energy.

51 [JDN] One might well think that only Einstein could seriously ask after the gravitational mass of such an oddity in gravitation theory as radiation enclosed in a massless, mirrored chamber. Yet Planck (1908, 4) had already asked exactly this question.

the system, taken as a whole, to be accorded a gravitational mass $\int P d\tau$, which corresponds with the energy E of the radiation.

Now instead of the radiation being enclosed in an empty box, I imagine it bounded
1. by the mirrored walls of a fixed shaft S,
2. by two vertically moveable, mirrored walls W_1 and W_2, which are firmly fixed to one another by a rod. (See figure 3.)

In this case the gravitational mass $\int P d\tau$ of the moving system amounts to only a third part of the value which arises for a box moving as a whole. Therefore, in raising the radiation against a gravitational field, one would have to expend only a third part of the work as in the case considered before, in which the radiation is enclosed in a box. This seems unacceptable to me.

Einstein's objection bears a little expansion. He has devised two means of raising and lowering some fixed quantity of radiation in a gravitational field. Notice that in either case the radiation by itself has no gravitational mass, since the trace of the stress-energy tensor of pure electromagnetic radiation vanishes. What introduces such a mass is the fact that the radiation is held within an enclosure upon which it exerts a pressure, so that the enclosure is stressed. Even though the members of the enclosure are assumed massless, it turns out that a gravitational mass must still be ascribed to them simply because they are stressed. The beauty of Einstein's argument is that the gravitational masses ascribed in each of the two cases can be inferred essentially without calculation.

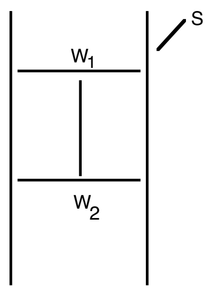

Figure 3: Rendering of Figure in Einstein's Text

In the first case, the radiation is moved in a mirrored box. The radiation and enclosing box form a complete static system. Therefore the gravitational mass of the box together with the radiation is given by the total energy of the radiation, $\int T_{44}^{\text{em}} dV$, in its rest frame, where I now write the stress-energy tensor of the electromagnetic radiation as $T_{\mu\nu}^{\text{em}}$. For ease of transition to the second case, it is convenient to imagine that each of the three pairs of opposing wall of the box is held in place *only* by a connecting rod, that the faces of the box are aligned with the x, y and z axes and that the disposition of the system is identical in these three directions. Each connecting rod will be stressed in reaction to the radiation pressure. I write $T_{\mu\nu}^{X}$ for the stress-energy tensor of the rod aligned in the x direction and the two stressed walls that this rod connects. $T^Y_{\mu\nu}$ and $T^Z_{\mu\nu}$ represent the other two corresponding systems. We can then infer directly from (37) that the gravitational mass of the entire system in its rest frame is proportional to

$$\int T^{\mathrm{em}}_{44} dV = \int T^{\mathrm{em}}_{\mu\mu} dV + \int T^X_{\mu\mu} dV + \int T^Y_{\mu\mu} dV + \int T^Z_{\mu\mu} dV = 3\int T^Z_{\mu\mu} dV.$$

The second equality follows from $T^{\mathrm{em}}_{\mu\mu} = 0$ and from the symmetry of the three axes, which entails

$$\int T^X_{\mu\mu} dV = \int T^Y_{\mu\mu} dV = \int T^Z_{\mu\mu} dV .$$

In Einstein's second case the radiation is trapped between sliding, mirrored baffles in a mirrored shaft aligned, let us say, in the z direction. The only component of the moveable system carrying a gravitation mass in this case will be the stressed rod and the stressed baffles it connects. Its gravitational mass in its rest frame will simply be proportional to the volume integral of the trace of its stress energy tensor. This integral is equal to $\int T^Z_{\mu\mu} dV$ and is one third of the corresponding integral for the first case, as Einstein claimed.

We now combine the two cases into a cycle. We lower the radiation inside the cube into the gravitational field, recovering some work, since the system has a gravitational mass. We then transfer the radiation into the baffle system and raise it. Only one third of the work released in the first step is needed to elevate the radiation because of the baffle system's reduced gravitational mass.[52] The mirrored cube and baffles are weightless once they are unstressed by the release of the radiation so they can be returned to their original positions. The cycle is complete with a net gain of energy.

That a theory should violate the conservation of energy is one of the most serious objections that Einstein could raise against it. Notice that he did not mention another possible objection that would derive directly from the vanishing of the trace of the stress-energy tensor of pure radiation. This vanishing entails that light cannot be deflected by a gravitational field. However, in 1913, prior to the experimental determination of this effect, Einstein could hardly have expected this last objection to have any force.

As devastating and spectacular as Einstein's objection was, he had at this time developed the unfortunate habit of advancing devastating arguments to prove conclusion he later wished to retract, see (Norton 1984, § 5). This objection to all theories of gravitation with a scalar potential proved to be another instance of his habit. Within a few months Einstein had endorsed (if not initiated) a most interesting escape.

52 Since the estimates of gravitational mass are made in the system's rest frames, these motions would have to be carried out infinitely slowly.

12. NORDSTRÖM'S SECOND THEORY

On July 24, 1913, Nordström submitted another version of his gravitation theory to *Annalen der Physik*, (Nordström 1913b). This version of the theory finally took proper notice of what Einstein had presented as obvious in his *Entwurf* paper. The only possible scalar that can represent gravitational source density is the trace of the stress-energy tensor and this choice, in conjunction with Laue's work on complete static systems, enables satisfaction of the requirement of equality of inertial and gravitational mass. Moreover the version of the requirement satisfied is an Einsteinian version in which the quantity of gravitational source is proportional to the total energy. This differs from the version embodied in Nordström's equation (16) in which the motion of a body in free fall is merely independent of its rest mass. Finally the theory offered an ingenious escape from Einstein's *Entwurf* objection. It turned out that the objection failed if one assumed that the proper length of a body would vary with the gravitational potential. This new version of this theory is sufficiently changed that it is now customarily known as Nordström's second theory.

There is room for interesting speculation on the circumstances under which Nordström came to modify his theory. In the introduction (p. 533) he thanked Laue and Einstein for identifying the correct gravitational source density. As we shall see, at two places in the paper (p. 544, 554), he also attributed arguments and results directly to Einstein without citation. Since I know of no place in which Einstein published these results, it seems a reasonable conjecture that Nordström learned of them either by correspondence or personal contact. That it was personal contact during a visit to Zurich at this time is strongly suggested by the penultimate line of the paper, which, in standard *Annalen der Physik* style, gives a place and date. It reads "Zurich, July 1913."[53] Since Nordström does thank both Laue and Einstein directly and in that order and since the wording of Einstein's *Entwurf* suggests a personal communication directly from Laue, we might conjecture also that there was similar direct contact between Laue and Nordström. Laue was also in Zurich at this time at the University of Zurich.

One cannot help but sense a somewhat sheepish tone in the introduction to (Nordström 1913b, 533), when he announced that this earlier presentation (Nordström 1913a) was "not completely unique" and that "the rest density of matter was defined in a fairly arbitrary way." In effect he was conceding that he had bungled the basic idea of his earlier (Nordström 1913a), that one had to take notice of the mechanics of stressed bodies in defining the gravitational source density and that this ought to be done in a way that preserved the equality of inertial and gravitational mass. Laue and Einstein were now telling him how he ought to have written that paper.

53 An entry in Ehrenfest's Diary ("I", NeLR, Ehrenfest Archive, Scientific Correspondence, ENB: 4–15) reveals a visit to Zurich by Nordström in late June. See (CPAE 4, 294–301), "Einstein on Gravitation and Relativity: the Collaboration with Marcel Grossmann."

12.1 The Identification of the Gravitational Source

The first task of Nordström (1913b) was to incorporate the new source density into his theory and this was tackled in its first section. The final result would be to define this density in terms of what he called the "elastic-material tensor" $T_{\mu\nu}$, which corresponded to the sum of Nordström's (1913a) material tensor and elastic stress tensor as given above in (31). Following Einstein and Laue, he ended up selecting $1/c^2$ times the negative[54] trace D of $T_{\mu\nu}$ as his source density v

$$\nu = \frac{1}{c^2}D = -\frac{1}{c^2}(T_{xx} + T_{yy} + T_{zz} + T_{uu}). \tag{38}$$

However unlike Laue and Einstein, that selection came at the conclusion of a fairly lengthy derivation. Nordström would show that the requirement of equality of inertial and gravitational mass in the case of a complete static system would force this choice of source density.

To begin, Nordström chose essentially the same field equation for the potential Φ and gravitational force density equations as in (Nordström 1913a):

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} + \frac{\partial^2\Phi}{\partial u^2} = g(\Phi)\nu\ , \tag{12''}$$

$$\mathfrak{K}^g_\mu = -g(\Phi)\nu\frac{\partial\Phi}{\partial x_\mu}\ . \tag{18'}$$

Here v remained the as yet undetermined gravitational source density. The important innovation was that the gravitation factor g was now allowed to vary as a function of the potential Φ. In an attempt to bring some continuity to the development of (Nordström 1913b) from (1913a), he recalled that in the former paper (p. 873, 878) he had foreshadowed the possibility that g might be a function of the inner constitution of bodies. Indeed the paper had closed with the speculation that such a dependence might enable the molecular motions of a falling body to influence its acceleration of fall, presumably as part of a possible escape from Einstein's original objection to his theory.

As it happened, however, the Φ dependence of g was introduced for an entirely different purpose in (Nordström 1913b). Einstein's version of the equality of inertial and gravitational mass required that the *total* energy of a system would be the measure of its gravitational source strength. This total energy would include the energy of the gravitational field itself. This requirement, familiar to us from Einstein's treatment of general relativity, leads to non-linearity of field equations. In Nordström's theory, this non-linearity would be expressed as a Φ dependence of g.

Nordström now turned to complete static systems, whose properties would yield not just the relation (38) but also a quite specific function for $g(\Phi)$. At this point

54 Since he retained his standard coordinate system of $x, y, z, u = ict$, the negative sign is needed to preserve the positive sign of v.

Nordström seemed able to give Laue a little taste of his own medicine. As I pointed out above, Laue's 1911 definition of a complete static system had excluded such systems as bodies rotating uniformly about their axis of symmetry. Nordström now made the obvious extension, defining what he called a "complete *stationary*" system. Curiously he made no explicit statement that his was a more general concept. Readers simply had to guess that his replacement of Laue's "static" by his "stationary" was no accident. Or perhaps they had to wait until Laue's (1917, 273) own concession that Nordström was first to point out the extension.

Nordström's complete stationary system had the following defining characteristics: it was a system of finite bodies for which a "justified"[55] reference system existed in which the gravitational field was static, that is, $\partial\Phi/\partial t = 0$. In particular in the relevant reference system, instead of Laue' s condition (28) which required the vanishing everywhere of the momentum density $\mathfrak{g}$, Nordström required merely that the *total* momentum $\mathfrak{G}$ vanished,

$$\mathfrak{G} = \int \mathfrak{g}\, dv = 0$$

where v is the volume of the body in its rest frame. The two illustrations Nordström gave—surely not coincidentally—were exactly two systems that Laue's earlier definition did not admit: a body rotating about its axis of symmetry and a fluid in stationary flow. Of course the first example was one of great importance to Nordström. It was precisely the example discussed in the final paragraphs of both (Nordström 1912 and 1913a). That such a system would fall more slowly than a non-rotating system was the substance of Einstein's original objection. Now able to apply the machinery of Laue's complete static systems to this example, Nordström could try to show that these rotating bodies did not fall slower in the new theory.

Nordström proceeded to identify the three stress-energy tensors which could contribute to the total energy of a complete stationary system. They were the "elastic-material tensor" $T_{\mu\nu}$, mentioned above; the "electromagnetic tensor" $L_{\mu\nu}$, which we would otherwise know as the stress-energy tensor of the electromagnetic field; and finally the "gravitation tensor" $G_{\mu\nu}$. This last tensor was the stress-energy tensor of the gravitational field itself. It had been identified routinely in (Nordström (1912, 1128; 1913a, 875). It was given by[56]

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55 In this context I read this to mean "inertial".

56 I continue to compress Nordström's notation. He did not use the Kronecker delta $\delta_{\mu\nu}$ and wrote individual expressions for G_{xx}, G_{xy}, etc. The derivation of (39) is brief and entirely standard. Writing ∂_μ for $\partial/\partial x_\mu$ we have, from substituting (12") into (18')

$$\begin{aligned}\mathfrak{K}^g_\mu &= -g(\Phi)\nu\partial_\mu\Phi = -(\partial_\nu\partial_\nu\Phi)\partial_\mu\Phi = -\left(\partial_\nu(\partial_\mu\Phi\partial_\nu\Phi) - \frac{1}{2}\partial_\mu(\partial_\lambda\Phi\partial_\lambda\Phi)\right) \\ &= -\partial_\nu\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\delta_{\mu\nu}(\partial_\lambda\Phi\partial_\lambda\Phi)\right) = -\partial_\nu G_{\mu\nu}\,.\end{aligned}$$

$$G_{\mu\nu} = \frac{\partial\Phi}{\partial x_\mu}\frac{\partial\Phi}{\partial x_\nu} - \frac{1}{2}\delta_{\mu\nu}\frac{\partial\Phi}{\partial x_\lambda}\frac{\partial\Phi}{\partial x_\lambda} \tag{39}$$

where Φ is the gravitational potential. Invoking Laue's basic result (29), which was also used by Einstein for the same end, Nordström could represent the total energy E_0 of a complete stationary system in its rest frame as the integral over all space of the sum of the traces of these three tensors

$$E_0 = -\int T + G + L \ dv \ .$$

However we have $L = 0$ and have written $T = -D$. Finally the integral of the trace G could be written in greatly simplified fashion if one assumed special properties for the complete stationary system. In particular, the gravitational potential Φ must approach the limiting constant value Φ_a at spatial infinity. From this and an application of Gauss' theorem, Nordström inferred that[57]

$$\int G \ dv = \int (\Phi - \Phi_a) g(\Phi) \nu \ dv \ . \tag{40}$$

Combining and noting that the inertial rest mass m of the system is E_0/c^2, we have

$$m = \frac{E_0}{c^2} = \frac{1}{c^2}\int \{D - (\Phi - \Phi_a) g(\Phi)\nu\} dv \ .$$

However we also have the total gravitational mass M_g of the system is

$$M_g = \int g(\Phi)\nu dv \ . \tag{41}$$

Nordström was now finally able to invoke what he calls "Einstein's law of equivalence," the equality of inertial and gravitational mass. Presumably viewing the completely stationary system from a great distance, one sees that it is a system with inertial mass m lying within a potential Φ_a so that we must be able to write

$$M_g = g(\Phi_a) m \ .$$

These last three equations form an integral equation which can only be satisfied identically if

57 Nordström's derivation of (40) seems to require the additional assumption that ν is non-zero only in some finite part of space. For a completely stationary system, the field equation reduces to the Newtonian $\partial_i \partial_i \Phi = g(\Phi)\nu$. If ν satisfies this additional assumption one can recover from it Nordström's result (his equation (5)), that $|\nabla\Phi| = |\partial_i \Phi| = \frac{1}{4\pi r^2}\int g(\Phi)\nu dv$, at large distances r from the system. This result seems to be needed to complete the derivation of (40).

$$D = g(\Phi)\nu\left\{\Phi - \Phi_a + \frac{c^2}{g(\Phi_a)}\right\}.$$

Finally Nordström required that this relation between D and ν be independent of Φ_a, from which the two major result of the analysis followed: first, $g(\Phi)$ is given by

$$g(\Phi) = \frac{c^2}{A + \Phi} \tag{42}$$

where A is a universal constant; second, he recovered the anticipated identity of the source density ν

$$\nu = \frac{1}{c^2}D = -\frac{1}{c^2}(T_{xx} + T_{yy} + T_{zz} + T_{uu}) . \tag{38}$$

The constant A in equation (42) is taken by Nordström to be an arbitrary additive gauge factor, corresponding to the freedom in Newtonian gravitation theory of setting an arbitrary zero point for the gravitational potential. However, in contrast with the Newtonian case, there is a natural gauge of $A = 0$ in which the equations are greatly simplified. Writing the potential that corresponds to the choice of $A = 0$ as Φ', the expression for $g(\Phi')$ is

$$g(\Phi') = \frac{c^2}{\Phi'} \tag{43}$$

and in this gauge one recovers a beautifully simple expression for the relationship between the total rest energy E_0, inertial rest mass m and gravitational mass M_g of a completely stationary system

$$E_0 = mc^2 = M_g\Phi'_a . \tag{44}$$

In particular, it contains exactly the Newtonian result that the energy of a system with gravitational mass M_g in a gravitational field with potential Φ'_a is $M_g\Phi'_a$. This was an improvement over Nordström's first theory were the closest corresponding result was (15), a dependence of mass on an exponential function of the potential.

12.2 Dependence of Lengths and Times on the Gravitational Potential

Satisfactory as these results were, Nordström had not yet answered the objection of Einstein's *Entwurf* paper. Indeed it is nowhere directly mentioned in Nordström's paper, although Nordström (1913b, 544) does cite the relevant part of the *Entwurf* paper to acknowledge Einstein's priority concerning the expression for the gravitational source density in equation (38). However at least the second, third and fourth parts of Nordström (1913b) are devoted implicitly to escaping from Einstein's objection. Nordström there developed a model of a spherical electron within his theory and showed that its size must vary inversely with the gravitational potential Φ'. It only becomes apparent towards the middle of Section 3 that this result must hold for the dimensions of all bodies and that this general result provides the escape from Einstein's objection. The general result is demonstrated by an argument attributed without any citation to Einstein. Since I know of no place where this argument was published by Einstein and since we know that Nordström was visiting Einstein at the time of submission of his paper, it is a reasonable supposition that he had the argument directly from Einstein in person. Since the general result appears only in the context of this argument, it is a plausible conjecture that the result, as well as its proof, is due to Einstein. Of course the successful recourse to an unusual kinematical effect of this type is almost uniquely characteristic of Einstein's work.

Einstein's argument takes the *Entwurf* objection and reduces it to its barest essentials. The violation of energy conservation inferred there depended solely on the behavior of the massless members of the systems that were oriented transverse to the direction of the field in which they moved. The transverse members lowered were stressed and thus were endowed with a gravitational mass so that work was recovered. The transverse members elevated were unstressed so that they had no gravitational mass and no work was required to elevate them. The outcome was a net gain in energy. Einstein's new argument considered this effect in a greatly simplified physical system. Instead of using radiation pressure to stress the transverse members, Einstein now just imagined a single transverse member—a non-deformable rod—tensioned between vertical rails. The gravitational mass of the rod is increased by the presence of these stresses. Since the rod expands on falling into the gravitational field, the rails must diverge and work must be done by the forces that maintain the tension in the rod. It turns out that this work done exactly matches the work released by the fall of the extra gravitational mass of the rod due to its stressed state. The outcome is that no net work is released. In recounting Einstein's argument, Nordström makes no mention of Einstein's *Entwurf* objection. This is puzzling. It is hard to imagine that he wished to avoid publicly correcting Einstein when the history of the whole theory had been a fruitful sequence of objections and correction and this correction was endorsed and even possibly invented by Einstein. In any case it should have been clear to a contemporary reader who understood the mechanism of Einstein's objection that the objection would be blocked by an analogous expansion of the systems as they fell into the gravitational field. Certainly Einstein (1913, 1253), a few months later, reported that his *Entwurf* objection failed because of the tacit assumption of the constancy of dimensions of the systems as they move to regions of different potential.[58]

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58 He gives no further analysis. It is clear, however, that as the mirrored box is lowered into the field and expands, work would be done by the radiation pressure on its walls. It will be clear from the ensuing analysis that this work would reduce the total energy of the radiation by exactly the work released by the lowering of the gravitational mass of the system. Thus when the radiation is elevated in the mirrored shaft the radiation energy recovered would be diminished by exactly the amount needed to preserve conservation of energy in the entire cycle.

Einstein's argument actually establishes that the requirement of energy conservation for such cycles necessitates a presumed isotropic expansion of linear dimensions to be in inverse proportion to the gravitational potential Φ'. Nordström's report of the argument reads:

> Herr Einstein has proved that the dependence in the theory developed here of the length dimensions of a body on the gravitational potential must be a general property of matter. He has shown that otherwise it would be possible to construct an apparatus with which one could pump energy out of the gravitational field. In Einstein's example one considers a non-deformable rod that can be tensioned moveably between two vertical rails. One could let the rod fall stressed, then relax it and raise it again. The rod has a greater weight when stressed than unstressed, and therefore it would provide greater work than would be consumed in raising the unstressed rod. However because of the lengthening of the rod in falling, the rails must diverge and the excess work in falling will be consumed again as the work of the tensioning forces on the ends of the rod.
>
> Let S be the total stress (stress times cross-sectional area) of the rod and l its length. Because of the stress, the gravitational mass of the rod is increased by[59]
>
> $$\frac{g(\Phi)}{c^2}Sl = \frac{1}{\Phi'}Sl .$$
>
> In falling [an infinitesimal distance in which the potential changes by $d\Phi'$ and the length of the rod by dl], this gravitational mass provides the extra work
>
> $$-\frac{1}{\Phi'}Sld\Phi'$$
>
> However at the same time at the ends of the rod the work is lost [to forces stressing the rod]. Setting equal these two expressions provides
>
> $$-\frac{1}{\Phi'}d\Phi' = \frac{1}{l}dl$$
>
> which yields on integration
>
> $$l\Phi' = \text{const.} \qquad [(45)]$$
>
> This, however, corresponds with [Nordström's] equation (25a) [the potential dependence of the radius of the electron].[60]

59 [JDN] To see this, align the x axis of the rest frame with the rod. The only non-vanishing component of $T_{\mu\nu}$ is T_{xx} which is the stress (per unit area) in the rod. Therefore, from (38) and (41), the gravitational mass

$$M_g = \int g(\Phi)\nu dv = -\frac{g(\Phi)}{c^2}Sl .$$

60 A footnote here considers a complication that need not concern us. It reads:
If the rod is deformable, in stressing it, some work will be expended and the rest energy of the rod will be correspondingly increased. Thereby the weight also experiences an increase, which provides the added work dA in falling. However, since in falling the rest energy diminishes, the work recovered in relaxing the rod is smaller than that consumed at the stressing and the difference amounts to exactly dA.

This result (45) was just one of a series of dependencies of basic physical quantities on gravitational potential. In preparation in Section 2 for his analysis of the electron, Nordström had already demonstrated that the inertial mass m of a complete stationary system varied in proportion to the external gravitational potential Φ':

$$\frac{m}{\Phi'} = \text{const.} \tag{46}$$

whereas the gravitational mass of the system M_g was independent of Φ'. The proof considered the special case of a complete stationary system for which the external field Φ'_a would be uniform over some sphere at sufficient distance from the system and directed perpendicular to the sphere. We might note that a complete stationary system of finite size within a constant potential field would have this property. He then imagined that the external field Φ'_a is altered by a slow displacement of yet more distant masses. He could read directly from the expression for the stress energy tensor of the gravitational field what was the resulting energy flow through the sphere enclosing the system and from this infer the alteration in total energy and therefore mass of the system. The result (46) followed immediately and from it the constancy of M_g through (44).[61] Calling on (45), (46) and other specific results in his analysis of the electron, Nordström was able to infer dependencies on gravitational potential for the stress S in the electron's surface, the gravitational source density ν and the stress tensor p_{ab}:

$$\frac{S}{\Phi'^3} = \text{const.} \quad \frac{\nu}{\Phi'^4} = \text{const.} \quad \frac{P_{ab}}{\Phi'^4} = \text{const.}$$

Finally, after these results had been established, the entire Section 5 was given to establishing that the time of a process T would depend on the potential Φ' according to

$$T\Phi' = \text{const.} \tag{47}$$

In particular it followed from this result that the wave lengths of spectral lines depends on the gravitational potential. Nordström reported that the wavelength of light from the sun's surface would be increased by one part in two million. He continued to report that "The same—possibly even observable—shift is given by several other new theories of gravitation." (p. 549) While he gave no citation, modern readers need hardly be told that all of Einstein's theories of gravitation from this period give this effect, including his 1907–1912 scalar theory of static fields, his *Entwurf* form of general relativity and the final 1915 generally covariant version of the theory. The very numerical value that Nordström reported—one part in two million—was first reported in Einstein's earliest publication on gravitation, (Einstein 1907a, 459).

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61 On pp. 545–46, he showed that, for a complete stationary system, the external potential Φ'_a varied in direct proportion to the potential Φ' at some arbitrary point within the system, so that Φ' and Φ'_a could be used interchangeably in expressing the proportionalities of the form of (45), (46) and (47).

Nordström devoted some effort to the proof of (47). He noted that it followed immediately from (45) and the constancy of the velocity of light for the time taken by a light signal traversing a a rod. Anxious to show that it held for other systems, he considered a small mass orbiting another larger mass M_1 in a circular orbit of radius r within an external potential field Φ'_a. The analysis proved very simple since the speed of the small mass, its inertial mass and the potential along its trajectory were all constant with time. He showed that its orbital period T satisfied

$$\frac{M_1 c^2}{4\pi r \Phi'_a} = \frac{4\pi^2 r^2}{T^2}. \tag{48}$$

As the potential Φ'_a varies, $r\Phi'_a$ remains constant according to (45). Therefore the equality requires that T vary in direct proportion to r from which it follows that T satisfies (47). Again he showed the same effect for the period of a simple harmonic oscillator of small amplitude.

12.3 Applications of Nordström's Second Theory: The Spherical Electron and Free Fall

So far we have seen the content of Nordström's second theory and how he established its coherence. The paper also contained two interesting applications of the theory. The first was an analysis of a spherical electron given in his Section 3. It turned out to yield an especially pretty illustration of the result that a gravitational mass is associated with a stress. For the *entire* gravitational mass of Nordström's electron proves to be due to an internal stress. The electron was modelled as a massless shell carrying a charge distributed on its surface. (See Appendix for details.) The shell must be stressed to prevent mechanical disintegration of the electron due to repulsive forces between parts of the charge distribution. The electric field does not contribute to the electron's gravitational mass since the trace of its stress-energy tensor vanishes. Since the shell itself is massless it also does not contribute to the gravitational mass when unstressed. However, when stressed in reaction to the repulsive electric forces, it acquires a gravitational mass which comprises the entire gravitational mass of the electron.

Nordström's model of the electron was not self contained in the sense that it required only known theories of electricity and gravitation. Like other theories of the electron at this time, it had to posit that the stability of the electron depended on the presence of a stress bearing shell whose properties were largely unknown. While one might hope that the attractive forces of gravitation would replace this stabilizing shell, that was not the case in Nordström's electron. Rather he was superimposing the effects of gravitation on a standard model of the electron.[62]

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62 We can see the Nordström had no real hope of eliminating this shell with gravitational attraction. For the electric field by itself generates no gravitational field in his theory. Another element must be present in the structure of the electron if gravitational forces are to arise.

The second illustration was an analysis in his final Section 7 of the motion in free fall of a complete stationary system. In particular, Nordström was concerned to determine just how close this motion was to the corresponding motion of a point mass. The results were not entirely satisfactory. He was able to show that complete stationary systems fell like point masses only for the case of a homogeneous external gravitational field, that is, one whose potential was a linear function of all four coordinates (x, y, z, u). He showed that a complete stationary system of mass m, falling with four velocity $\mathfrak{B}_\mu$ in a homogeneous external field Φ_a, obeys equations of motion

$$-g(\Phi'_a)m\frac{\partial\Phi_a}{\partial x_\mu} = \frac{d}{d\tau}m\mathfrak{B}_\mu \tag{49}$$

which corresponded to the equations (13) for a point mass in his first theory, excepting the added factor $g(\Phi'_a)$. Allowing that the mass m varies inversely with Φ'_a through equation (46), it follows that explicit mention of the mass m can be eliminated from these equations of motion which become[63]

$$-c^2\frac{\partial}{\partial x_\mu}\ln\Phi'_a = \frac{d}{d\tau}\mathfrak{B}_\mu - \mathfrak{B}_\mu\frac{d}{d\tau}\ln\Phi_a\,.$$

Nordström's concern was clearly still Einstein's original objection to this first theory recounted above in Section 7. A body rotating about its axis of symmetry could form a complete stationary system. He could now conclude that such a body would fall exactly as if it had no rotation, contrary, as he noted, to the result of his earlier theory. Also, he concluded without further discussion that molecular motions would have no influence on free fall. However, the vertical acceleration of free fall would continue to be slowed by its initial velocity according to (19) of his first theory.

We might observe that stresses would play a key role in the cases of the rotating body and the kinetic gas. The rotating body would be stressed to balance centrifugal forces and the walls containing a kinetic gas of molecules would be stressed by the forces of the gas pressure. These stresses would add to the gravitational mass of the spinning body and the contained gas allowing them to fall independently of their internal motions. No such compensating stresses would be present in the case of a point mass or a complete stationary system projected horizontally, so they would fall slower due to their horizontal velocity.

Thus, while Nordström's theory finally satisfied the requirement of equality of inertial and gravitational mass in Einstein's sense, it still did not satisfy the requirement that all bodies fall alike in a gravitational field. This Einstein (1911, § 1) called "Galileo's principle," elsewhere (1913, 1251) citing it as the fact of experience supporting the equality of inertial and gravitational mass. Galileo's principle held only under rather restricted conditions: the system must be in vertical fall and in a homogeneous field.[64] At least, however, he could report that Einstein had extended the

63 I have corrected Nordström's incorrect "+" to "–" on the right-hand side.

result to systems that were not complete stationary systems. He had shown that the *average* acceleration of an elastically oscillating system accorded with (19). Since this last result is nowhere reported in Einstein's publications, we must assume that he had it directly from Einstein.

Finally, in the course of his exposition, Nordström could note that the mass dependence on Φ' of relation (46) now replaces the corresponding condition (15) of his first theory. The new variable factor of $g(\Phi') = c^2/\Phi'$ in the in the equation of motion (49) causes (14) to be replaced by $\frac{mc^2}{\Phi'}\frac{d\Phi'}{d\tau} = c^2\frac{dm}{d\tau}$, which integrates to yield (46).

13. EINSTEIN FINALLY APPROVES: THE VIENNA LECTURE OF SEPTEMBER 1913

In September 1913, Einstein attended the 85th Congress of the German Natural Scientists and Physicians. There he spoke on the subject of the current state of the problem of gravitation, giving a presentation of his new *Entwurf* theory and engaging in fairly sharp dispute in discussion. A text for this lecture with ensuing discussion was published in the December issue of *Physikalische Zeitschrift* (Einstein 1913). Einstein made clear (p. 1250) his preference for Nordström's theory over other gravitation theories, including Abraham's and Mie's. Nordström's latest version of his gravitation theory was the only competitor to Einstein's own new *Entwurf* theory satisfying four requirements that could be asked of such gravitation theories:

1. "Satisfaction of the conservation law of momentum and energy;"
2. "Equality of inertial and gravitational masses of closed systems;"
3. Reduction to special relativity as a limiting case;
4. Independence of observable natural laws from the absolute value of the gravitational potential.

What Einstein did not say was that the satisfaction of 1. and 2. by Nordström's theory was due in significant measure to Einstein's pressure on Nordström and Einstein's own suggestions.

Einstein devoted a sizeable part of his lecture to Nordström's theory, giving a self-contained exposition of it in his Section 3. That exposition was a beautiful illustration of Einstein's ability to reduce the complex to its barest essentials and beyond. He simplified Nordström's development in many ways, most notably:

64 One might think that this would give Einstein grounds for rejecting the competing Nordström theory in favor of his own *Entwurf* theory. At the appropriate place, however, Einstein (1913, 1254) did *not* attack Nordström on these grounds. Perhaps that was for the better since it eventually turned out that general relativity fared no better. In general relativity, for example, a rotating body falls differently, in general, from a non-rotating body. See (Papapetrou 1951; Corinaldesi and Papapetrou 1951).

- Einstein selected the natural gauge (43) for the potential Φ', writing the resulting potential without the prime as φ.
- Einstein eradicated the implicit potential dependence of the mass m in (46), using a new mass m which did not vary with potential. This meant that Einstein's m coincided with the gravitational mass, not the inertial mass of a body.

To begin, Einstein used as the starting point the "Hamiltonian" equation of motion (36) which he had first recommended in Section 7 of his *Entwurf* paper. Using coordinates $(x_1, x_2, x_3, x_4) = (x, y, z, ict)$, he wrote this equation of motion of a mass m as

$$\delta\{\int H dt\} = 0 \tag{50}$$

where

$$H = -m\varphi\frac{d\tau}{dt} = m\varphi\sqrt{c^2 - q^2} .$$

Here q is the coordinate three-speed and τ is the Minkowski interval given by

$$d\tau^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2 . \tag{51}$$

Since Einstein varied the three spatial coordinates of the particle trajectory x_i, the resulting equation of motion governed the three-velocity $\dot{x}_i = \frac{dx_i}{dt}$

$$\frac{d}{dt}\left\{m\varphi\frac{\dot{x}_i}{\sqrt{c^2 - q^2}}\right\} - m\frac{\partial\varphi}{\partial x_i}\sqrt{c^2 - q^2} = 0 .$$

It also followed that the momentum (increased by a multiplicative factor c) I_i and the conserved energy E were given by

$$I_i = m\varphi\frac{\dot{x}_i}{\sqrt{c^2 - q^2}} , \quad E = m\varphi\frac{c^2}{\sqrt{c^2 - q^2}} .$$

In particular, one could read directly from these formulae that the inertial mass of a body of mass m at rest is given by $m\varphi/c^2$ and that its energy is $m\varphi$.

Einstein then introduced the notion that had rescued Nordström's theory from his own recent attack: directly measured lengths and times might not coincide with those given by the Minkowski line element (51). He called the former quantities "natural" and indicated them with a subscript 0. He called the latter "coordinate" quantities. The magnitude of the effect was represented by a factor ω which would be a function of φ and was defined by

$$d\tau_0 = \omega d\tau . \tag{52}$$

Allowing for the dependence of energy on φ and the effects of the factor ω, Einstein developed an expression for the stress-energy tensor $T_{\mu\nu}$ of "flowing, incoherent matter"—we would now say "pressureless dust"—in terms of its natural mass density ρ_0 and the corresponding gravitational force density k_μ:

$$T_{\mu\nu} = \rho_0 c\varphi\omega^3 \frac{dx_\mu}{dt}\frac{dx_\nu}{dt}, \quad k_\mu = -\rho_0 c\varphi\omega^3 \frac{\partial\varphi}{\partial x_\mu}.$$

The two quantities were related by the familiar conservation law

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = k_\mu .$$

The next task was to re-express this conservation law in terms of the trace $T_{\sigma\sigma}$ of the stress-energy tensor. Mentioning Laue's work, Einstein remarked that this quantity was the only choice for the quantity measuring the gravitational source density. For the special case of incoherent matter, $T_{\sigma\sigma} = -\rho_0 c\varphi\omega^3$, so that the conservation law took on a form independent of the special quantities involved in the case of incoherent matter flow

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = T_{\sigma\sigma}\frac{1}{\varphi}\frac{\partial\varphi}{\partial x_\mu}, \tag{53}$$

Einstein announced what was really an assumption: this form of the law governed arbitrary types of matter as well.

This general form of the conservation law allowed Einstein to display the satisfaction by the theory of the second requirement he had listed. That was the equality of inertial and gravitational masses of closed systems. His purpose in including the additional words "closed systems" now became clear. In effect he meant by them Laue's complete static systems. His demonstration of the satisfaction of this result was admirably brief but damnably imprecise, compared to the careful attention Nordström had lavished on the same point. Einstein simply assumed that he had a system over whose spatial extension there was little variation in the φ term $\frac{1}{\varphi}\frac{\partial\varphi}{\partial x_\mu} = \frac{\partial \log\varphi}{\partial x_\mu}$ on the right hand side of (53). An integration of (53) over the spatial volume v of such a system revealed that the four-force acting on the body is

$$\frac{\partial \log\varphi}{\partial x_\mu}\int T_{\sigma\sigma}dv = \frac{\partial \log\varphi}{\partial x_\mu}\int T_{44}dv ,$$

where the terms in T_{11}, T_{22}, and T_{33} were eliminated by Laue's basic result (29). Since $\int T_{44}dv$ is the negative of the total energy of the system, Einstein felt justified to conclude: "Thereby is proven that the weight of a closed system is determined by its total measure [of energy]." Einstein's readers might well doubt this conclusion and suspect that the case of constant φ considered was a special case that may not be representative of the general case. Fortunately such readers could consult (Nordström 1913b) for a more precise treatment.

In his lecture, Einstein was seeking to give an exposition of both Nordström's and his new theory of gravitation and reasons for deciding between them. Thus we might anticipate that he had to cut corners somewhere. And that place turned out to be the singular novelty of Nordström's theory in 1913, the potential dependence of lengths and times. His introduction of this effect and concomitant retraction of his *Entwurf* objection was so brief that only someone who had followed the story closely and read the report of Einstein's argument in (Nordström 1913b) could follow it. Virtually all he had to say lay in a short paragraph (p. 1253):

> Further, equation [(53)] allows us to determine the function [ω] of φ left undetermined from the physical assumption that no work can be gained from a static gravitational field through a cyclical process. In § 7 of my jointly published work on gravitation with Herr Grossmann I generated a contradiction between the scalar theory and the fundamental law mentioned. But I was there proceeding from the tacit assumption that ω = const[ant]. The contradiction is resolved, however, as is easy to show, if one sets[65]
>
> $$l = \frac{l_0}{\omega} = \frac{\text{const.}}{\varphi}$$
>
> or
>
> $$\omega = \text{const.} \cdot \varphi \,. \qquad [(54)]$$
>
> We will give yet a second substantiation for this stipulation later.

That second substantiation followed shortly, immediately after Einstein had given the field equation of Nordström's theory. He considered two clocks. The first was a "light clock," a rod of length l_0 with mirrors at either end and a light signal propagating in a vacuum and reflected between them. The second was a "gravitation clock," two gravitationally bound masses orbiting about one another at constant distance l_0. He gave no explicit analysis of these clocks. His only remark on their behavior was that their relative speed is independent of the absolute value of the gravitational potential, in accord with the fourth of the requirements he had laid out earlier for gravitation theories. This, he concluded, "is an indirect confirmation of the expression for ω given in equation [(54)]."

Einstein's readers would have had to fill in quite a few details here. Clearly the dependence of l_0 on the potential would cause the period of the light clock also to vary according to (47). But readers would also need to know of the analysis of the gravitation clock given by (Nordström 1913b) which led to (48) above and the same dependence on potential for the clock's period. Thus the dependence of both periods is the same so that the relative rate of the two clocks remains the same as the external potential changes. Had this result been otherwise, the fourth requirement would have been violated. That it was not presumably displays the coherence of the theory and thereby provides the "*indirect* confirmation." Curiously Einstein seems not to be

65 l is defined earlier as the length of a body. This retraction is also mentioned more briefly (p. 261) in the addendum to the later printing of (Einstein and Grossmann 1913) in the *Zeitschrift*.

making the obvious point that his equations (52) and (54) together yield the same potential dependence for periodic processes as follows from the behavior of these two clocks—or perhaps he deemed that point too obvious to mention.

The final component of the theory was its field equation. Recalling that "Laue's scalar" $T_{\sigma\sigma}$ must enter into this equation, Einstein simply announced it to be

$$-\kappa T_{\sigma\sigma} = \varphi \frac{\partial^2}{\partial x_\tau^2}\varphi . \tag{55}$$

It became apparent that the additional factor of φ on the right hand side was included to ensure compatibility with the conservation of energy and momentum.[66] To display this compatibility he noted that stress-energy tensor $t_{\mu\nu}$ of the gravitational field is

$$t_{\mu\nu} = \frac{1}{\kappa}\left\{\frac{\partial\varphi}{\partial x_\mu}\frac{\partial\varphi}{\partial x_\nu} - \frac{1}{2}\delta_{\mu\nu}\left(\frac{\partial\varphi}{\partial x_\tau}\right)^2\right\} .$$

This tensor satisfies the equalities

$$T_{\sigma\sigma}\frac{1}{\varphi}\frac{\partial\varphi}{\partial x_\mu} = -\frac{1}{\kappa}\frac{\partial\varphi}{\partial x_\mu}\frac{\partial^2\varphi}{\partial x_\tau^2} = -\frac{\partial t_{\mu\nu}}{\partial x_\nu}$$

The first depends on substitution of $T_{\sigma\sigma}$ by the field equation and the second holds identically. Substituting into the conservation law (53) yields an expression for the joint conservation of gravitational and non-gravitational energy momentum,[67]

$$\frac{\partial}{\partial x_\nu}(T_{\mu\nu} + t_{\mu\nu}) = 0 .$$

All that remained for Einstein was to give his reasons for not accepting Nordström's theory. In our time, of course, the theory is deemed an empirical failure because it does not predict any deflection of a light ray by a gravitational field and does not explain the anomalous motion of Mercury. However in late 1913, there had been no celebrated eclipse expeditions and Einstein's own *Entwurf* theory also did not explain the anomalous motion of Mercury. Thus Einstein's sole objection to the theory was not decisive, although we should not underestimate its importance to Einstein. According to Nordström's theory, the inertia of a body with mass m was $m\varphi/c^2$.

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66 Although Einstein does not make this point, it is helpful to divide both sides by φ and look upon $T_{\sigma\sigma}/\varphi$ as the gravitational source density. The trace $T_{\sigma\sigma}$ represents the mass-energy density and division by φ cancels out this density's φ dependence to return the gravitational mass density.

67 As Michel Janssen has repeatedly emphasized to me, Einstein's analysis is a minor variant of the method he described and used to generate the field equations of his *Entwurf* theory. Had Einstein begun with the identity mentioned, the expression for $t_{\mu\nu}$ and the conservation law (53), a reversal of the steps of Einstein's argument would generate the field equation. For further discussion of Einstein's method, see (Norton 1995).

Therefore, as the gravitational field in the neighborhood of the body was intensified by, for example, bringing other masses closer, the inertia of the body would actually decrease. This was incompatible with Einstein's idea of the "relativity of inertia" according to which the inertia of a body was caused by the remaining bodies of the universe, the precursor of what he later called "Mach's Principle." This deficiency enabled Einstein to ask after the possibility of extending the principle of relativity to accelerated motion, to see the real significance of the equality of inertial and gravitational mass in his principle of equivalence (which was not satisfied by Nordström's theory) and to develop his *Entwurf* theory.

14. EINSTEIN AND FOKKER: GRAVITATION IN NORDSTRÖM'S THEORY AS SPACETIME CURVATURE

It was clear by the time of Einstein's Vienna lecture that Nordström's most conservative of approaches to gravitation had led to a something more than a conservative Lorentz covariant theory of gravitation, for it had become a theory with kinematical effects very similar to those of Einstein's general theory of relativity. Gravitational fields would slow clocks and alter the lengths of rods. All that remained was the task of showing just how close Nordström's theory had come to Einstein's theory. This task was carried out by Einstein in collaboration with a student of Lorentz', Adriaan D. Fokker, who visited Einstein in Zurich in the winter semester of 1913–1914 (Pais 1982, 487). Their joint (Einstein and Fokker 1914), submitted on February 19, 1914, was devoted to establishing essentially one result, namely, in modern language, Nordström's theory was actually the theory of a spacetime that was only conformal to a Minkowski spacetime with the gravitational potential the conformal factor, so that the presence of a gravitational field coincided with deviations of the spacetime from flatness. That, of course, was not how Einstein and Fokker described the result. Their purpose, as they explained in the title and introduction of the paper, was to apply the new mathematical methods of Einstein's *Entwurf* theory to Nordström's theory. These methods were the "absolute differential calculus" of Ricci and Levi-Civita (1901). They enabled a dramatic simplification of Nordström's theory. It will be convenient here to summarize the content of the theory from this new perspective as residing in three basic assumptions:

I. Spacetime admits preferred coordinate systems $(x_1, x_2, x_3, x_4) = (x, y, z, ct)$ in which the spacetime interval is given by

$$ds^2 = \Phi^2(dx^2 + dy^2 + dz^2 - c^2dt^2) \tag{56}$$

and in which the trajectory of point masses in free fall is given by

$$\delta\int ds = 0\ .$$

That such a characterization of the spacetime of Nordström's theory is possible is implicit in Einstein's Vienna lecture. In fact, once one knows the proportionality of

ω and φ, the characterization can be read without calculation from Einstein's expression (52) for the natural proper time and the equation of motion (50). Einstein and Fokker emphasized that the preferred coordinate systems are ones in which the postulate of the constancy of the velocity of light obtains. For, along a light beam $ds^2 = 0$, so that

$$dx^2 + dy^2 + dz^2 - c^2dt^2 = 0 .$$

We see here in simplest form the failure of the theory to yield a deflection of a light beam in a gravitational field. This failure is already evident, of course, from the fact that a light beam has no gravitational mass since the trace of its stress-energy tensor vanishes.

II. The conservation of gravitational and non-gravitational energy momentum is given by the requirement of the vanishing of the covariant divergence of the stress-energy tensor $T_{\mu\nu}$ for non-gravitational matter. At this time, Einstein preferred to write this condition as[68]

$$\sum_{\nu} \frac{\partial \mathfrak{T}_{\sigma\nu}}{\partial x_{\nu}} = \frac{1}{2} \sum_{\mu\nu\tau} \frac{\partial g_{\mu\nu}}{\partial x_{\sigma}} \gamma_{\mu\tau} \mathfrak{T}_{\tau\nu} ,$$

since they could interpret the term on the right hand side as representing the gravitational force density.

Noting, as Einstein and Fokker did on pp. 322–23, that the $T_{\mu\nu}$ of the Vienna lecture corresponds to the tensor density $\mathfrak{T}_{\mu\nu}$ of the new development, they evaluated this conservation law in the preferred coordinate systems of I. It yielded the form of the conservation law (53) of the Vienna lecture.

Finally Einstein and Fokker turned to the field equation which was to have the form

$$\Gamma = k\mathfrak{T}$$

where κ is a constant. The quantity $\mathfrak{T}$ had to be a scalar representing material processes. In the light of the earlier discussion, we know there was only one viable choice, the trace of the stress-energy tensor $T = \frac{1}{\sqrt{-g}} \sum_{\tau} \mathfrak{T}_{\tau\tau}$. For the quantity Γ, which must be constructed from the metric tensor and its derivatives, they reported that the researches of mathematicians allowed only one quantity to be considered, the full contraction of the Riemann-Christoffel tensor (ik, lm) of the fourth rank, where

68 Here Einstein had not yet begun to use modern notational conventions. Summation over repeated indices is not implied. All indices are written as subscript so that $\gamma_{\mu\nu}$ is the fully contravariant form of the metric, which we would now write as $g^{\mu\nu}$. $\mathfrak{T}_{\mu\nu}$ is the mixed tensor density which we could now write as $\sqrt{-g}T^{\nu}_{\mu}$.

they allowed i, k, l and m to vary over 1, 2, 3 and 4. This assumed that the second derivative of $g_{\mu\nu}$ enters linearly into the equation. Therefore we have:

III. The gravitational field satisfies the field equation which asserts the proportionality of the fully contracted Riemann-Christoffel tensor and the trace of the stress energy tensor

$$\sum_{iklm} \gamma_{ik}\gamma_{lm}(ik, lm) = \kappa \frac{1}{\sqrt{-g}} \sum_{\tau} \mathfrak{T}_{\tau\tau} \, .$$

Evaluation of this field equation in the preferred coordinate systems of I. yields the field equation (55) of the Vienna lecture.

Einstein and Fokker were clearly and justifiably very pleased at the ease with which the methods of the *Entwurf* theory had allowed generation of Nordström's theory. In the paper's introduction they had promised to show that (p. 321)

> ... one arrives at Nordström's theory instead of the Einstein-Grossmann theory if one makes the single assumption that it is possible to choose preferred reference systems in such a way that the principle of the constancy of the velocity of light obtains.

Their concluding remarks shine with the glow of their success when they boast that (p. 328)

> ... one can arrive at Nordström's theory from the foundation of the principle of the constancy of the velocity of light through purely formal considerations, i.e. without assistance of further physical hypotheses. Therefore it seems to us that this theory earns preference over all other gravitation theories that retain this principle. From the physical stand point, this is all the more the case, as this theory achieves strict satisfaction of the equality of inertial and gravitational mass.

Of course Einstein retained his objection that Nordström's theory violates the requirement of the relativity of inertia.[69] The new formulation gives us vivid demonstration of this failure: the disposition of the preferred coordinate systems of I. will be entirely unaffected by the distribution of matter in spacetime. Einstein must then surely have been unaware that it would prove possible to give a generally covariant formulation of Nordström's theory on the basis of Weyl's work (Weyl 1918). The requirement that the preferred coordinate systems of I. exist could be replaced by the generally covariant requirement of the vanishing of the conformal curvature tensor. This formal trick, however, does not alter the theory's violation of the relativity of inertia and the presence of preferred coordinate systems in it.

There remained a great irony in Einstein and Fokker's paper, which their readers would discover within two short years. While the existence of preferred coordinate systems was held against the Nordström theory, Einstein's own *Entwurf* theory was

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69 As we know from lecture notes taken by a student, Walter Dallenbach, (EA 4 008, 41-42), Einstein in his teaching at the ETH in Zurich at this time included the claim that one arrives at the Nordström theory merely by assuming there are specialized coordinate system in which the speed of light is constant.There he remarks that this theory violates the relativity of inertia.

not itself generally covariant and would not be until November 1915, when Einstein would disclose the modern field equations to the Prussian Academy. Einstein and Grossmann (1913) had settled upon gravitational field equations which were not generally covariant. We now know that the generally covariant field equations of the completed general theory of relativity can be derived by means of the Riemann-Christoffel tensor through an argument very similar to the one used to arrive at the generally covariant form of the field equation of the Nordström theory. Einstein and Grossmann had considered and rejected this possibility in § 4.2 of Grossmann's part of their joint paper. The obvious ease with which consideration of the Riemann-Christoffel tensor led to the field equation of Nordström's theory clearly gave Einstein an occasion to rethink that rejection. For Einstein and Fokker's paper concluded with the tantalizing remark that the reasons given in Grossmann's § 4 of their joint paper against such a connection did not withstand further examination. Whatever doubt this raised in Einstein's mind seem to have subsided by March 1914, at which time he reported in a letter to this confidant Michele Besso that the "general theory of invariants functioned only as a hindrance" in construction of his system (Speziali 1972, 53).

Thus the conservative path struck by Nordström and Einstein led not just to the connection between gravitation and spacetime curvature but to the first successful field equation which set an expression in the Riemann-Christoffel curvature tensor proportional to one in the stress-energy tensor of matter.

15. WHAT EINSTEIN KNEW IN 1912

Einstein and Fokker's characterization in 1914 of the Nordström theory gives us a convenient vantage point from which to view Einstein's theory of 1912 for static gravitational fields. In particular we can see clearly that this theory already contained many of the components that would be assembled to form Nordström's theory. Indeed we shall see that Einstein's theory came very close to Nordström's theory. However we shall also see that a vital component was missing—the use of the stress-energy tensor and Laue's work on complete static systems. This component enables a scalar Lorentz covariant theory of gravitation to satisfy some version of the requirement of the equality of inertial and gravitational mass. We must already suspect that Einstein was unaware of this possibility prior to his August 1912 move to Zurich for his July 1912 response to Abraham (Einstein 1912d), quoted in Section 4 above, purports to show that no Lorentz covariant theory of gravitation could satisfy this requirement.

Einstein (1912a, 1912b) was the fullest development of a relativistic theory of static gravitational fields based on the principle of equivalence and in which the gravitational potential was the speed of light c. By Einstein's own account the following year (Einstein and Grossmann 1913, I, § 1, § 2), the theory was actually a theory of a spacetime with the line element

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2dt^2 \tag{57}$$

where c is now a function of x, y and z and behaves as a gravitational potential. Einstein (1912a, 360) offered the field equation

$$\nabla^2 c = kc\sigma \tag{58}$$

where k is a constant and σ the rest density of matter.[70] What Einstein did not mention in his *Entwurf* reformulation of the 1912 theory was that this field equation corresponded to the generally covariant field equation

$$R = \frac{k}{2}T$$

where R is the fully contracted Riemann-Christoffel tensor and T the trace of the stress–energy tensor, in the case of an unstressed, static matter distribution. This is exactly the field equation of Nordström's theory!

This field equation (58) had an extremely short life, for in (Einstein 1912b, § 4), a paper submitted to *Annalen der Physik* on March 23, 1912, just a month after February 26, when he had submitted (Einstein 1912a), he revealed the disaster that had befallen his theory and would lead him to retract this field equation. Within the theory the force density $\mathfrak{F}$ on a matter distribution σ at rest is

$$\mathfrak{F} = -\sigma \mathrm{grad} c\ .$$

Einstein conjoined this innocuous result with the field equation (58) and applied it to a system of masses at rest held together in a rigid massless frame within a space in which c approached a constant value at spatial infinity. He concluded that the total gravitational force on the frame

$$\int \mathfrak{F} d\tau = -\int \sigma \mathrm{grad} c d\tau = -\frac{1}{k}\int \frac{\nabla^2 c}{c} \mathrm{grad} c d\tau$$

in general does not vanish. That is, the resultant of the gravitational forces exerted by the bodies on one another does not vanish. Therefore the system will set itself into motion, a violation of the equality of action and reaction, as Einstein pointed out. In effect the difficulty lay in the theory's failure to admit a gravitational field stress tensor, for the gravitational force density $\mathfrak{F}$ is equal to the divergence of this tensor. Were the tensor to be definable in Einstein's theory, that fact alone, through a standard application of Gauss' theorem, would make the net resultant force on the system vanish.[71]

Einstein then proceeded to consider a number of escapes from this disaster. The second and third escapes involved modifications to the force law and the field equation. The former failed but the latter proved workable. Einstein augmented the source density σ of (58) with a term in c:

70 The factor of c on the right hand side of this otherwise entirely classical equation is introduced in order to leave c undetermined by a multiplicative gauge factor rather than an additive one.

$$\nabla^2 c = k\left\{c\sigma + \frac{1}{2k}\frac{\text{grad}^2 c}{c}\right\}.$$

The extra term was constructed to allow the formation of a gravitational field stress tensor and the conclusion that there would be no net force on the system of masses. Einstein was especially pleased to find that this extra term proved to represent the gravitational field energy density so that the source term of the field equation was now the total energy density of the system, gravitational and non-gravitational.[72]

For our purposes what is most interesting is the first escape that Einstein considered and rejected. Mentioning vaguely "results of the old theory of relativity," he considered the possibility that the stressed frame of the system might have a gravitational mass. That possibility was dismissed however with an argument that is surprising to those familiar with his work of the following year: that possibility would violate the equality of inertial and gravitational mass! Einstein considered a box with mirrored walls containing radiation of energy E. He concluded from his theory that, if the box were sufficiently small, the radiation would exert a net force on the walls of the box of $-E\text{grad}c$. He continued (Einstein 1912b, 453):

> This sum of forces must be equal to the resultant of forces which the gravitational field exerts on the whole system (box together with radiation), if the box is massless and if the circumstance that the box walls are subject to stresses as a result of the radiation pressure does not have the consequence that the gravitational field acts on the box walls. Were the latter the case, then the resultant of the forces exerted by the gravitational field on the box (together with its contents) would be different from the value $-E\text{grad}c$, *i.e.* the gravitational mass of the system would be different from E.

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71 Writing $t_{im} = \frac{1}{kc}\left(\partial_i c \partial_m c - \frac{1}{2}\delta_{im}(\partial_n c \partial_n c)\right)$ for the quantity that comes closest to the stress tensor, we have the following in place of the standard derivation of the stress tensor (analogous to the derivation of (39)). Substituting field equation (58) into the expression for $\mathfrak{F}$, we recover:

$$\mathfrak{F}_i = -\sigma\partial_i c = -\frac{1}{kc}\partial_m c \partial_m c \partial_i c = -\partial_m t_{im} - \frac{1}{2k}\frac{\partial_m c \partial_m c}{c^2}\partial_i c\ .$$

The first term of the final sum is a divergence which would vanish by Gauss' theorem when integrated over the space containing the masses of the frame, leaving no net force. The problem comes from the second term, which is present only because of the factor of c on the source side of the field equation (58). In this integration it will not vanish in general, leaving the residual force on the masses. The need to eliminate this second term also dictates the precise form of the modification to the field equation that Einstein ultimately adopted. When the field equation source σ was augmented to become $\sigma + \frac{1}{2k}\frac{\partial_m c \partial_m c}{c^2}$, this second term no longer arose in the above expression for $\mathfrak{F}_i$.

72 However Einstein was disturbed to find that the new field equation only allowed his principle of equivalence to apply to infinitesimally small parts of space. See (Norton 1985, § 4.2, § 4.3).

Einstein could not have written this were he aware of the relevant properties of "Laue's scalar" T. As Einstein himself showed the following year, the use of T as the gravitational source density in exactly this example of radiation enclosed in a mirrored cavity allowed one to infer *both* that the walls of the cavity acquired a gravitational mass because of their stressed state and that the gravitational mass of the entire system was given by its total energy. We must then take Einstein at his word and conclude that he learned of these properties of T from Laue. Presumably this means after his move to Zurich in August 1912 where Laue also was, and after completion of his work on his scalar theory of static gravitational fields in 1912.

Had Einstein been aware of these results earlier in 1912, they would probably not have pleased him in the long run. To begin, he did believe at the time of writing the *Entwurf* paper that the selection of T as the gravitational source density in a scalar theory of gravitation led to a contradiction with the conservation of energy. Had he seen past this to its resolution in the gravitational potential dependence of lengths he would have arrived at a most remarkable outcome: his theory of 1912 would have become exactly Nordström's final theory! As we saw above, his first field equation of 1912 was already equivalent to Nordström's final field equation in covariant terms. His equation of motion for a mass point was already the geodesic equation for a spacetime with the line element (57). This line element already entailed a dependence of times on the gravitational potential. The consistent use of Laue's scalar T as a source density would finally have led to a similar dependence for spatial length so that the line element (57) would be replaced by Nordström's (56). Since the expressed purpose of Einstein's 1912 theory was to extend the principle of relativity, this out come would not have been a happy one for Einstein. For his path would have led him to a theory which entailed the existence of coordinate systems in which the speed of light was globally constant. That is, the theory had resurrected the special coordinate systems of special relativity.

16. THE FALL OF NORDSTRÖM'S THEORY OF GRAVITATION

Revealing as Einstein and Fokker's formulation of the theory had been, Nordström himself clearly did not see it as figuring in the future development of his theory. Rather, Nordström embedded his 1913 formulation of his gravitation theory in his rather short lived attempts to generate a unified theory of electricity and gravitation within a five dimensional spacetime (Nordström 1914c, 1914d, 1915). Other work on the theory in this period was devoted to developing a clearer picture of the behavior of bodies in free fall and planetary motion according to the theory. Behacker (1913) had computed this behavior for Nordström's first theory and (Nordström 1914a) performed the same service for his second theory. In both cases the behavior demanded by the theories was judged to be in complete agreement with experience.

Nordström also had to defend his theory from an attack by Gustav Mie. Mie had made painfully clear in the discussion following Einstein's Vienna lecture of 1913 (published in *Physikalische Zeitschrift*, *14*, 1262–66) that he was outraged over Einstein's failure even to discuss Mie's own theory of gravitation in the lecture. Einstein explained that this omission derived from the failure of Mie's theory to satisfy the requirement of the equality of inertial and gravitational mass. Mie counterattacked with a two part assault (Mie 1914) on Einstein's theory. In an appendix (§10) Mie turned his fire upon Nordström's theory, claiming that it violated the principle of energy conservation. Nordström's (1914b) response was that Mie had erroneously inferred the contradiction within Nordström's theory by improperly importing a result from Mie's own theory into the derivation. Laue (1917, 310–13) pointed to errors on both sides of this dispute.

However it was not Mie's theory that led to the demise of Nordström's theory. Rather it was the rising fortunes of Einstein's general theory of relativity. Einstein completed the theory in a series of papers submitted to the Prussian Academy in November 1915. Within a few years, with the success of Eddington's eclipse expedition, Einstein had become a celebrity and his theory of gravitation eclipsed all others. One of the papers from that November 1915 (Einstein 1915) reported the bewitching success of the new theory in explaining the anomalous motion of Mercury. This success set new standards of empirical adequacy for gravitation theories. Prior to this paper, the pronouncements of a gravitation theory on the minutia of planetary orbits were not deemed the ultimate test of a new theory of gravitation. Einstein's own *Entwurf* theory failed to account for the anomalous motion of Mercury. Yet this failure is not mentioned in Einstein's publications from this period and one cannot even tell from these publications whether he was then aware of it. Thus the treatment in (Nordström 1914a) of the empirical adequacy of his theory to observed planetary motions was entirely appropriate by the standards of 1914. He showed that his theory predicted a very slow retardation of the major axis of a planet's elliptical orbit. Computing this effect for the Earth's motion he found it to be 0.0065 seconds of arc per year, which could be dismissed as "very small in relation to the astronomical perturbations [due to other planets]" (p.1109) Thus he could proceed to the overall conclusion (p. 1109) that

> ... the laws derived for [free] fall and planetary motion are in the *best* agreement with experience [my emphasis]

Standards had changed so much by the time of Laue's (1917) review article on the Nordström theory that even motions much smaller than the planetary perturbations were decisive in the evaluation of a gravitation theory. Einstein's celebrated 43 seconds of arc per century advance of Mercury's perihelion is less than a tenth of the perihelion motion due to perturbations from the other planets. Laue (p. 305) derived a formula for the predicted retardation—not advance—of a planet's perihelion. Without even bothering to substitute values into the formula he lamented

> Therefore the perihelion moves opposite to the sense of rotation of the orbit. In the case of Mercury, the impossibility of explaining its perihelion motion with this calculation lies already in this difference of sign concerning the perihelion motion.

Through this period, Nordström's theory had its sympathizers and the most notable of these was Laue himself.[73] He clearly retained this sympathy when he wrote the lengthy review article, (Laue 1917). Einstein's theory had become so influential by this time that Laue introduced the review with over four pages of discussion of Einstein's theory (pp. 266–70). That discussion conceded that Einstein's theory had attracted the most adherents of any relativistic gravitation theory. It also contained almost two pages of continuous and direct quotation from Einstein himself, as well as discussion of the epistemological and empirical foundations of Einstein' s theory. His discussion was not the most up-to-date, for he reported Einstein's *Entwurf* 0.84 seconds of arc deflection for a ray of starlight grazing the sun, rather than the figure of 1.7 of the final theory of 1915. All this drove to the conclusion that there were no decisive grounds for accepting Einstein's theory and provided Laue with the opportunity to review a gravitation theory based on special relativity, Nordström's theory, which he felt had received less attention than it deserved.

The fall of Nordström's theory was complete by 1921. By this time even Laue had defected. In that year he published a second volume on general relativity to accompany his text on special relativity (Laue 1921). On p.17, he gave a kind appraisal of the virtues and vices of his old love, Nordström's theory. However he was firm in his concluding the superiority of Einstein's theory because of the failure of Nordström's theory to yield any gravitational light deflection—a defect, he urged, that must trouble any Lorentz covariant gravitation theory. Laue never lost his affection for the theory and years later took the occasion of Einstein's 70th birthday to recall the virtues of Nordström's theory (Laue 1949). The theory's obituary appeared in Pauli's encyclopedic distillation of all that was worth knowing in relativity theory (Pauli 1921, 144). He pronounced authoritatively

> The theory solves in a logically quite unexceptionable way the problem sketched out above, of how to bring the Poisson equation and the equation of motion of a particle into a Lorentz-covariant form. Also, the energy-momentum law and the theorem of the equality of inertial and gravitational mass are satisfied. If, in spite of this, Nordström's theory is not acceptable, this is due, in the first place, to the fact that it does not satisfy the principle of *general* relativity (or at least not in a simple and natural way ...). Secondly, it is in contradiction with experiment: it does not predict the bending of light rays and gives the displacement of the perihelion of Mercury with the wrong sign. (It is in agreement with Einstein's theory with regard to the red shift.)

He thereby rehearsed generations of physicists to come in the received view of Nordström's theory and relieved them of the need to investigate its content any further.

73 In a letter of October 10, 1915, to Wien, Mie had identified Laue as an adherent of Nordström's theory, explaining it through Laue's supposed failure to read anything else! I am grateful to John Stachel for this information.

17. CONCLUSION

The advent of the general theory of relativity was so entirely the work of just one person— Albert Einstein—that we cannot but wonder how long it would have taken without him for the connection between gravitation and spacetime curvature to be discovered. What would have happened if there were no Einstein? Few doubt that a theory much like special relativity would have emerged one way or another from the researches of Lorentz, Poincaré and others. But where would the problem of relativizing gravitation have led? The saga told here shows how even the most conservative approach to relativizing gravitation theory still did lead out of Minkowski spacetime to connect gravitation to a curved spacetime. Unfortunately we still cannot know if this conclusion would have been drawn rapidly without Einstein's contribution. For what led Nordström to the gravitational field dependence of lengths and times was a very Einsteinian insistence on just the right version of the equality of inertial and gravitational mass. Unceasingly in Nordström's ear was the persistent and uncompromising voice of Einstein himself demanding that Nordström see the most distant consequences of his own theory.

APPENDIX: NORDSTRÖM'S MODEL OF THE ELECTRON

Nordström's (1913b) development of his second theory contains (§ 3) a model of the electron which accounts for the effect of gravitation. The electron is modelled as a massless spherical shell of radius a carrying charge e distributed uniformly over its surface.[74] Three types of matter are present: an electric charge and its field; the shell stressed to balance the repulsive electric forces between different parts of the charge distribution; and the gravitational field generated by all three types of matter. See Figure 4. Taking each in turn, we have

74 "Rational" units of charge are used, which means, in effect, that the electrostatic field equation is $\Delta\Psi = -\rho$, for charge density ρ.

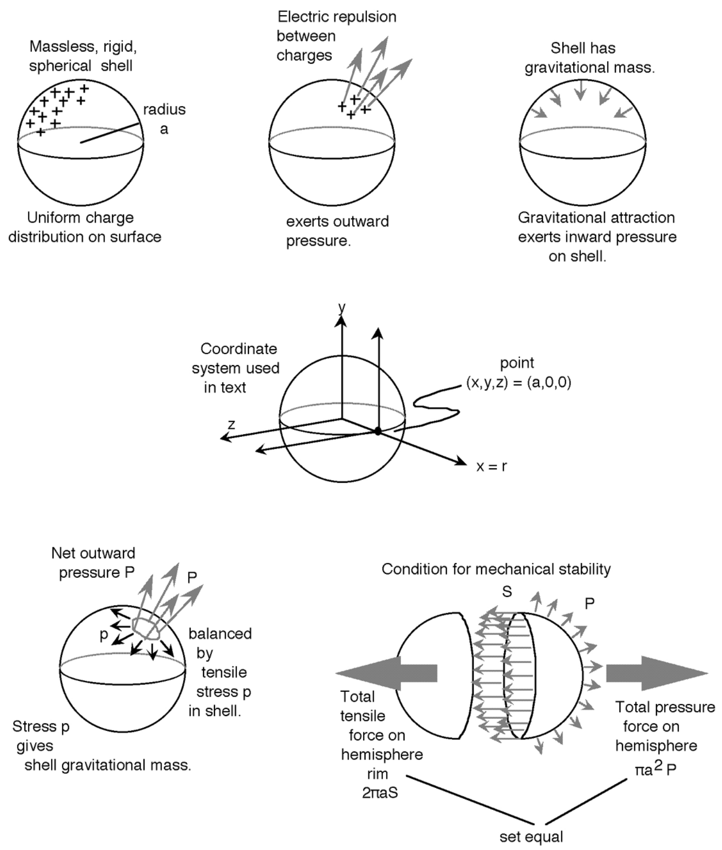

Figure 4: Nordström's Model of the Electron

Electric Charge and its Field

Using familiar results of electrostatics in the rational system of units, the electric charge e generates an electric potential Ψ at radius r from the center of the shell, for the case of $r \geq a$, which satisfies

$$\Psi = \frac{e}{4\pi r}, \quad \frac{\partial\Psi}{\partial r} = -\frac{e}{4\pi r^2}.$$

The latter value is all that is required to compute the Maxwell stress tensor at an arbitrary point on the shell which is representative of all its points due the rotational symmetry of the shell. We choose convenient coordinates $(x_1, x_2, x_3) = (x, y, z)$ for this point. We set the origin at the center of the shell, align the x axis with a radial arm and consider a point on the surface of the shell at which $(x, y, z) = (a, 0, 0)$. Writing ∂_i for $(\partial/\partial x_i)$, we have that the Maxwell stress tensor is[75]

$$L_{ik} = -\left(\partial_i \Psi \partial_k \Psi - \frac{1}{2}\delta_{ik}(\delta_m \Psi \partial_m \Psi)\right)$$

$$= -\begin{bmatrix} \frac{e^2}{32\pi^2 r^4} & 0 & 0 \\ 0 & -\frac{e^2}{32\pi^2 r^4} & 0 \\ 0 & 0 & -\frac{e^2}{32\pi^2 r^4} \end{bmatrix}.$$

We read directly from the coefficients of this tensor that the charges of the shell (at position $r = a$) are subject to an outwardly directed pressure of magnitude $e^2/32\pi a^4$ which seeks to cause the shell to explode radially outwards. That is, these charges are subject to a net electric force density given by the negative divergence of this stress tensor, $\partial_k L_{ik}$. With $r = a$, this force density is of magnitude $4e^2/32\pi^2 a^5$ directed radially outward.

Gravitational Field

The stresses in the shell will generate a gravitational field. For the moment, we shall write the total gravitational mass as M_g and note that it must be distributed uniformly over the shell. Since the source gravitational mass is all located in the shell over which the gravitational potential is constant, the field equation and stress tensor of the gravitational field reduce to the analogous equations of electrostatics, excepting a sign change. Thus the gravitational potential for $r \geq a$ satisfies

$$\Phi = -\frac{M_g}{4\pi r} + \Phi_a, \quad \frac{\partial \Phi}{\partial r} = \frac{M_g}{4\pi r^2},$$

where Φ_a is the external gravitational potential. Choosing the same point and coordinate system as in the analysis of the electric field, we find that the gravitational field stress tensor, as given by the spatial parts of the gravitational stress-energy tensor (39) is

75 The nonstandard minus sign follows the convention Nordström used in paper of requiring that (force density) =—(divergence of stress tensor). See (Nordström 1913b, 535, eq. 7).

$$G_{ik} = \begin{bmatrix} \frac{M_g^2}{32\pi^2 r^4} & 0 & 0 \\ 0 & -\frac{M_g^2}{32\pi^2 r^4} & 0 \\ 0 & 0 & -\frac{M_g^2}{32\pi^2 r^4} \end{bmatrix}.$$

This reveals an inwardly directed pressure $M_g^2/(32\pi^2 a^4)$ which seeks to implode the shell. That is, the shell is subject to a net gravitational force density given by the negative divergence of this stress tensor, $\partial_k T_{ik}$. With $r = a$, this force density is of magnitude $4M_g^2/32\pi^2 a^5$ directed radially inwards.

Stressed Shell

The combined effect of both electric and gravitational forces is a net outward pressure on the shell of magnitude

$$P = \frac{e^2 - M_g^2}{32\pi^2 a^4}. \tag{59}$$

Mechanical stability is maintained by a tensile stress p in the shell. At the point considered above in the same coordinate systems, this stress will correspond to a stress tensor T_{ik} given by

$$T_{ik} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix},$$

where p will have a negative value. If this tensile stress is integrated across the thickness of the shell, we recover the tensile force S per unit length active in the shell

$$S = -\int p\,ds.$$

The condition for mechanical stability is[76]

$$2S = aP. \tag{60}$$

76 This standard result from the theory of statics can be derived most easily, as Nordström points out, by considering the pressure forces due to P acting on a hemisphere of the shell. A simple integration shows this force is $\pi a^2 P$. This force must be balanced exactly by the tensile force S along the rim of the hemisphere. That rim is of length $2\pi a$, so the total force is $2\pi aS$. Setting $2\pi aS = \pi a^2 P$ entails the result claimed.

Computation of gravitational mass M_g and inertial mass m of the electron

The as yet undetermined gravitational mass M_g of the electron is now recovered by combining the results for the three forms of matter. The source density v is determined by the stress-energy tensor $T_{\mu\nu}$ through equation (38). By assumption, there is no energy associated with the tensile stress in the shell in its rest frame. Thus in a rest frame $T_{uu} = 0$. The spatial components of $T_{\mu\nu}$ are given by the stress tensor T_{ik} above. Therefore

$$\nu = -\frac{1}{c^2}(T_{xx} + T_{yy} + T_{zz} + T_{uu}) = -\frac{1}{c^2}2p .$$

We can now recover the gravitational mass M_g from (41) by integrating over the shell

$$M_g = \int g(\Phi)\nu dv = \frac{g(\Phi)}{c^2}8\pi a^2 S .$$

We now substitute S in this expression with the condition (60) for mechanical stability and thence for P with the condition (59). By means of (42), we can also express $g(\Phi)$ in terms of $g(\Phi_a) = g_a$ using

$$g(\Phi) = \frac{g_a}{1 + \frac{g_a}{c^2}(\Phi - \Phi_a)} = \frac{g_a}{1 - \frac{g_a}{c^2}\frac{M_g}{4\pi a}} .$$

After some algebraic manipulation, we recover an implicit expression for M_g

$$M_g = \frac{g_a}{c^2}\frac{e^2 + M_g^2}{8\pi a} .$$

Since this gravitational mass M_g of this complete stationary system resides in an external potential Φ_a, the total mass of the system satisfies $M_g = g_a m$, so that we have for the rest mass m and rest energy E_0 of the electron

$$m = \frac{E_0}{c^2} = \frac{e^2 + M_g^2}{8\pi c^2 a} . \tag{61}$$

As Nordström points out of this final result of § 3 of his paper is an extremely satisfactory one. The total energy of his electron is made up solely of the sum of an electric component $e^2/8\pi a$ and a gravitational component $M_g^2/8\pi a$. These two components agree exactly with the corresponding classical values. This agreement is not a foregone conclusion since the gravitational mass of the electron arises in an entirely non-classical way: it derives from the fact that the electron shell is stressed. Presumably this agreement justifies Nordström's closing remark in his § 3, "Thus the expression found for m contains a verification of the theory."

In his § 4, Nordström proceeded to use his expression (61) for the mass m of an electron to introduce the dependence of length on gravitational potential. In accordance with (46), derived in his § 2, the mass m must vary in proportion to the external field Φ'_a in the appropriate gauge. However it was not clear how one could recover this same variability from the quantities in the expression (61) for m. He had found in § 2 that M_g is independent of the gravitational potential and he asserted that the same held for e according to the basic equations of electrodynamics. Thus he concluded that the radius a of the electron must vary with gravitational potential according to (45). He then turned to Einstein' s more general argument for (45).

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REFERENCES

Abraham, Max. 1909. "Zur elektromagnetischen Mechanik." *Physikalische Zeitschrift*, 10: 737–41.

———. 1910. "Die Bewegungsgleichungen eines Massenteilchens in der Relativtheorie." *Physikalische Zeitschrift*, 11: 527–31.

———. 1912a. "Zur Theorie der Gravitation." *Physikalische Zeitschrift*, 13: 1–4.

———. 1912b. "Das Elementargesetz der Gravitation." *Physikalische Zeitschrift*, 13: 4–5.

———. 1912c. "Relativität und Gravitation. Erwiderung auf einer Bemerkung des Hrn. A. Einstein." *Annalen der Physik*, 38: 1056–58.

———. 1912d. "Nochmals Relativität und Gravitation. Bemerkung zu A. Einsteins Erwiderung." *Annalen der Physik*, 39: 444–48.

———. 1912e. "Eine neue Gravitationstheorie." *Archiv der Mathematik und Physik* (3), XX: 193–209.

Behacker, M. 1913. "Der freie Fall and die Planetenbewegung in Nordströms Gravitationstheorie." *Physikalische Zeitschrift*, 14: 989–992.

Corinaldesi, E. and A. Papapetrou, A. 1951. "Spinning Test Particles in General Relativity II." *Proceedings of the Royal Society, London*, *A*209: 259–68.

CPAE 2: John Stachel, David C. Cassidy, Jürgen Renn, and Robert Schulmann (eds.), *The Collected Papers of Albert Einstein*. Vol. 2. *The Swiss Years: Writings, 1900–1909*. Princeton: Princeton University Press, 1989.

CPAE 4: Martin J. Klein, A. J. Kox, Jürgen Renn, and Robert Schulmann (eds.), *The Collected Papers of Albert Einstein*. Vol. 4. *The Swiss Years: Writings, 1912–1914*. Princeton: Princeton University Press, 1995.

Ehrenfest, Paul. 1907. "Die Translation deformierbarer Elektronen und der Flächensatz." *Annalen der Physik*, 23: 204–205.

Einstein, Albert. 1907a. "Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen." *Jahrbuch der Radioaktivität und Elektronik*, (1907) 4: 411–462; (1908) 5: 98–99.

———. 1907b. "Bemerkungen zu der Notiz von Hrn Paul Ehrenfest: 'Die Translation deformierbarer Elektronen und der Flächensatz'." *Annalen der Physik*, 23: 206–208.

———. 1907c. "Über die vom Relativitätsprinzip gefordete Trägheit der Energie." *Annalen der Physik*, 23: 371–384.

———. 1911. "Über den Einfluss der Schwerkraft auf die Ausbreitung des Lichtes." *Annalen der Physik*, 35: 898–908.

———. 1912a. "Lichtgeschwindigkeit und Statik des Gravitationsfeldes." *Annalen der Physik*, 38: 355–69.

———. 1912b. "Zur Theorie des Statischen Gravitationsfeldes." *Annalen der Physik*, 38: 443–58.

———. 1912c. "Gibt es eine Gravitationswirkung die der elektrodynamischen Induktionswirkung analog ist?" *Vierteljahrsschrift für gerichtliche Medizin und öffentliches Sanitätswesen*, 44: 37–40.

———. 1912d. "Relativität und Gravitation. Erwiderung auf eine Bemerkung von M. Abraham." *Annalen der Physik*, 38: 1059–64.

———. 1912e. "Bemerkung zu Abrahams vorangehender Auseinandersetzung 'Nochmals Relativität und Gravitation'." *Annalen der Physik*, 39: 704.

———. 1913. "Zum gegenwärtigen Stande des Gravitationsproblems." *Physikalische Zeitschrift*, 14: 1249–1262.

———. 1915. "Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie." *Königlich Preussische Akademie der Wissenschaften* (Berlin). *Sitzungsberichte*. 1915: 831–39.

———. 1933. "Notes on the Origin of the General Theory of Relativity." In *Ideas and Opinions*, 285–290. Translated by Sonja Bargmann. New York: Crown, 1954.

———. 1949. *Autobiographical Notes*. Open Court, 1979.

Einstein, Albert and Adriaan D. Fokker. 1914. "Die Nordströmsche Gravitationstheorie vom Standpunkt des absoluten Differentialkalküls." *Annalen der Physik*, 44: 321–28.

Einstein, Albert and Marcel Grossmann. 1913. *Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation*. Leipzig: B.G.Teubner (separatum); with addendum by Einstein in *Zeitschrift für Mathematik und Physik*, 63(1913): 225–61.

Holton, Gerald. 1975. "Finding Favor with the Angel of the Lord. Notes towards the Psychobiographical Study of Scientific Genius." In Yehuda Elkana (ed.), *The Interaction between Science and Philosophy*. Humanities Press.

Isaksson, Eva. 1985. "Der finnische Physiker Gunnar Nordström und sein Beitrag zur Entstehung der allgemeinen Relativitätstheorie Albert Einsteins." *NTM-Schriftennr. Gesch. Naturwiss., Technik, Med.* Leipzig, 22, 1: 29–52.

Janssen, Michel. 1995. "A Comparison between Lorentz's Ether Theory and Einstein's Special Theory of Relativity in the Light of the Experiments of Trouton and Noble." PhD dissertation, University of Pittsburgh.

Laue, Max. 1911a. "Zur Dynamik der Relativitätstheorie." *Annalen der Physik*, 35: 524–542.

———. 1911b. *Das Relativitätsprinzip*. Braunschweig: Friedrich Vieweg und Sohn.

———. 1911c. "Ein Beispiel zur Dynamik der Relativitätstheorie." *Verhandlungen der Deutschen Physikalischen Gesellschaft*, 1911: 513–518.

———. 1912. "Zur Theorie des Versuches von Trouton und Noble." *Annalen der Physik*, 38: 370–84.

———. 1917. "Die Nordströmsche Gravitationstheorie." *Jahrbuch der Radioaktivität und Elektronik*, 14: 263–313.

———. 1921. *Die Relativitätstheorie*. Vol. 2: *Die allgemeine Relativitätstheorie und Einsteins Lehre von der Schwerkraft*. Braunschweig: Friedrich Vieweg und Sohn.

———. 1949. "Zu Albert Einsteins 70-tem Geburtstag." *Rev. Mod. Phys.*, 21: 348–49.

Lewis, Gilbert N. and Richard C. Tolman. 1909. "The Principle of Relativity, and Non-Newtonian Mechanics." *Philosophical Magazine*, 18: 510–523.

Liu, Chuang. 1991. *Relativistic Thermodynamics: Its History and Foundation*. Ph.D. Dissertation, University of Pittsburgh.

Maxwell, James C. 1864. "A dynamical theory of the electromagnetic field." In W. D. Niven (ed.), *The Scientific Papers of James Clerk Maxwell*. Cambridge University Press, 1890. Reprinted 1965, New York: Dover, 526–597.

Mie, Gustav. 1914. "Bemerkungen zu der Einsteinschen Gravitationstheorie. I and II." *Physikalische Zeitschrift*, 14: 115–122; 169–176.

Minkowski, Hermann. 1908. "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern." *Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, Nachrichten* 1908: 53–111; In *Gesammelte Abhandlung von Hermann Minkowski* Vol.2, Leipzig, 1911, 352–404. Reprinted New York: Chelsea. Page citations from this edition.

———. 1909. "Raum und Zeit." *Physikalische Zeitschrift*, 10: 104–111. In *Gesammelte Abhandlung von Hermann Minkowski* Vol.2, Leipzig, 1911, 431–444. Reprinted New York: Chelsea (page citations from this edition). Translated as "*Space and Time*," in H.A.Lorentz et al., *Principle of Relativity*, 1923, 75–91. Reprinted in 1952, New York: Dover.

Misner, Charles W., Kip S. Thorne, and John A. Wheeler. 1973. *Gravitation*. San Francisco: Freeman.

Nordström, Gunnar. 1909. "Zur Elektrodynamik Minkowskis." *Physikalische Zeitschrift*, 10: 681–87.

———. 1910. "Zur elektromagnetischen Mechanik." *Physikalische Zeitschrift*, 11: 440–45.

———. 1911. "Zur Relativitätsmechanik deformierbar Körper." *Physikalische Zeitschrift*, 12: 854–57.

———. 1912. "Relativitätsprinzip und Gravitation." *Physikalische Zeitschrift*, 13: 1126–29.

———. 1913a. "Träge und schwere Masse in der Relativitätsmechanik." *Annalen der Physik*, 40: 856–78.

———. 1913b. “Zur Theorie der Gravitation vom Standpunkt des Relativitätsprinzip.” *Annalen der Physik*, 42: 533–54.

———. 1914a. “Die Fallgesetze und Planetenbewegungen in der Relativitätstheorie.” *Annalen der Physik*, 43: 1101–10.

———. 1914b. “Über den Energiesatz in der Gravitationstheorie.” *Physikalische Zeitschrift*, 14: 375–80.

———. 1914c. “Über die Möglichkeit, das elektromagnetische Feld und das Gravitationsfeld zu vereinigen.” *Physikalische Zeitschrift*, 15: 504–506.

———. 1914d. “Zur Elektrizitäts- und Gravitationstheorie.” *Ofversigt af Finska Vetenskaps-Societetens Förhandlingar*, 57, (1914–1915), Afd. A, N:o.4: 1–15.

———. 1915. “Über eine Mögliche Grundlage einer Theorie der Materie,” *Ofversigt af Finska Vetenskaps-Societetens Förhandlingar*, 57, (1914–1915), Afd. A, N:o.28: 1–21.

Norton, John D. 1984. “How Einstein found his Field Equations: 1912–1915.” *Historical Studies in the Physical Sciences*, 14: 253–316. Reprinted 1989 in Don Howard and John Stachel (eds.), *Einstein and the History of General Relativity* (*Einstein Studies* vol. 1). Boston/Basel/Berlin: Birkhäuser, 101–159.

———. 1985. “What was Einstein’ s Principle of Equivalence?” *Studies in History and Philosophy of Science*, 16, 203–246. Reprinted 1989 in Don Howard and John Stachel (eds.), *Einstein and the History of General Relativity.* (*Einstein Studies* vol. 1.) Boston/Basel/Berlin: Birkhäuser, 3–47. (Page citations from the former.)

———. 1992. “The Physical Content of General Covariance.” In J. Eisenstaedt and A. Kox (eds.), *Studies in the History of General Relativity: Einstein Studies*, vol. 3. Boston/Basel/Berlin: Birkhäuser.

———. 1993. “Einstein and Nordström: Some Lesser-Known Thought Experiments in Gravitation.” In J. Earman, M. Janssen and J. D. Norton (eds.), *The Attraction of Gravitation: New Studies in the History of General Relativity.* (*Einstein Studies* vol. 5.) Boston/Basel/Berlin: Birkhäuser, 3–29.

———. 1995. “Eliminative Induction as a Method of Discovery: How Einstein Discovered General Relativity.” In J. Leplin (ed.), *The Creation of Ideas in Physics: Studies for a Methodology of Theory Construction*. Dordrecht: Kluwer, 29–69.

Pais, A. 1982. *Subtle is the Lord ...: The Science and the Life of Albert Einstein*. Oxford: Clarendon.

Papapetrou, A. 1951. “Spinning Test Particle in General Relativity I.” *Proceedings of the Royal Society, London*, A209, 248–58.

Pauli, Wolfgang. 1921. “Relativitätstheorie.” In *Encyklopädie der mathematischen Wissenschaften, mit Einschluss an ihrer Anwendung*. Vol. 5, *Physik*, Part 2. Arnold Sommerfeld (ed.). Leipzig: B.G. Teubner, 1904–1922, 539–775. [Issued November 15, 1921.] English translation *Theory of Relativity.* With supplementary notes by the author. G. Field. Translated in 1958, London: Pergamon. (Citations from the Pergamon edition.)

Planck, Max. 1908. “Zur Dynamik Bewegte Körper.” *Annalen der Physik*, 26: 1–34.

Poincaré, Henri. 1905. “Sur la Dynamique de l’ Électron.” *Comptes Rendus des Séances de l’Academie des Sciences*, 140: 1504–1508.

———. 1906. “Sur la Dynamique de l’ Électron,” *Rendiconti del Circolo Matematico di Palermo*, 21: 129–75.

Ricci, Gregorio and Tullio Levi-Civita. 1901. “Méthodes de Calcul Différentiel Absolu et leurs Application.” *Math. Ann.*, 54: 125–201. Reprinted 1954 in T. Levi-Civita, *Opere Matematiche*, vol. 1, Bologna, 479–559.

Rohrlich, Fritz. 1960. “Self-Energy and Stability of the Classical Electron.” *American Journal of Physics*, 28: 639–43.

Sommerfeld, Arnold. 1910. “Zur Relativitätstheorie I. Vierdimensionale Vektoralgebra.” *Annalen der Physik*, 32: 749–776; “Zur Relativitätstheorie II. Vierdimensionale Vektoranalysis.” *Annalen der Physik*, 33: 649–89.

Speziali, Pierre (ed.). 1972. *Albert Einstein-Michele Besso: Correspondance 1903–1955*. Paris: Hermann.

Tolman, Richard C. 1934. *Relativity, Thermodynamics and Cosmology.* Oxford: Oxford University Press; Dover reprint, 1987.

Trouton, Frederick T. and H. R. Noble. 1903. “The Mechanical Forces Acting on a Charged Condensor moving through Space.” *Philosophical Transactions of the Royal Society of London*, 202: 165–181.

Weyl, Hermann. 1918. “Reine Infinitesimal Geometrie.” *Mathematische Zeitschrift*, 2: 384–411.

Wheeler, John A. 1979. “Einstein’s Last Lecture.” In G. E. Tauber (ed.), *Albert Einstein’s Theory of General Relativity.* New York: Crown, 187–190.