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Lotteries, bookmaking and ancient randomizers: Local and global analyses of chance

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ABSTRACT

A local analysis of a chance system recovers the properties of the totality by accumulating the chance properties of its components. A global analysis considers only the properties of the chance system taken as a whole. It can sometimes provide an easier demonstration of the fitness for purpose of chance systems. Such is the case for the operators of lotteries and for bookmakers at a racetrack. Physical randomizers akin to dice were used extensively in antiquity for gambling, drawing lots and in divination. There is no ancient account of the fitness of these randomizers for these purposes and no ancient theory of the individual chances of outcomes. A global analysis can establish their fitness for purpose. If the rules of gambling and lot drawing are such that everyone has an equal opportunity, they are fair, independently of the individual, local chances. The randomizers are fit for the purposes of divination in so far as it is believed that the oracle has no control on the outcomes of the randomizers. In asking why some earlier culture did not discover probability theory, we presume incorrectly an inevitability to probability theory. The better question is why any culture found the theory at all.

1. Introduction

My goal in this paper is to delineate an approach to chance systems that is an older, weaker alternative to the modern approach. We now most commonly analyze chance systems locally. That is, we identify the individual outcomes comprising some chance system; we determine the probabilities of these outcomes; and we then accumulate them to infer to the chance properties of the system overall, that is, at the global level. In the alternative approach, we restrict the analysis to the global level. The analysis recognizes that individual outcomes in the system are chancy. However, it proceeds so that the results of the analysis are independent of the chances of each of the individual outcomes. Global analysts may have no theory of the chances of individual outcomes; or they may recognize that probabilities or some other chance notion can be applied to individual outcomes. Which is the case does not matter, since any such local chances do not figure in the global analysis.

One goal of chance analysis is to show that the chance system is fit for the purpose intended. I will focus on this goal. Controlled trials provide an illustration of local analysis. A physical randomizer can be used to determine which participants are assigned to the test and control groups. Local analysis in the probability theory tells us that this use of the randomizer is fit for the purposes of the study, since it makes it probable that the members of the two groups are uncorrelated in various properties.

Here I will show how global analysis establishes that the fitness for purpose of some chance systems. A reflective lottery player, buying a ticket, would now likely use the local approach. From the numbers of tickets sold and prizes offered, they would infer the probability of winning and compare that with the value of the prizes. Section 2 recounts how a lottery operator would consider the lottery globally and be indifferent to the individual probabilities. Profitability is assured if the tally of fees from the totality of tickets sold is greater than the totality of prizes offered. The lottery is fit for the operator's purpose of raising funds.

We might imagine that individual bookmakers at horse-racing tracks set their odds using a superior knowledge of the probability of each horse winning. However, as is related in Section 3, traditional bookmaking uses a less risky, global strategy. By adjusting the odds in response to bets made, a balanced book can be secured and an assured profit realized.

These local and global notions will be used to clarify a long-standing problem. Ancient people used physical randomizers extensively. They played games and gambled with dice and knucklebones. They drew lots to make decisions. Who inherits which plots of land? Who takes which position of authority? They used randomizers to divine the future and to

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discern the will of the gods. How can these uses of physical randomizers be seen to be fit for their purposes?

Section 4-7 below will develop what I believe is the simplest and likely most pertinent answer: the ancients developed no local theory of chance since they had no need of it. Their purposes were served by a global analysis in which individual probabilities are not useful. Kidd (2020), in his analysis of ancient and later gambling, has also determined that ancient gamblers approached the chance properties of their games globally.

Section 4 below will review the ancient use of physical randomizers in games and gambling, in lotteries and in divination. Given the pervasive ancient use of physical randomizers, I had expected to find ancient narratives explaining the concepts of chance implemented in the devices and why these devices were fit for their purposes. Section 5 will review extant remarks of ancient commentators on physical randomizers. They are too meager to show how the ancients came to believe that their randomizers were fit for their purposes. We are left to speculate.

Section 6 offers two conjectures. First, ancient gambling and lot drawing conforms, at the global level, with a broad notion of equality. All the conditions, including the rules of the games, are such that everyone has identical opportunities for loss or gain. Thus, they are fit for their purpose in the sense of treating everyone equally or fairly. That different faces of astragali—ancient randomizers—arise with different chances would be critical for local analysis. It is unimportant in the global analysis since the differences affect all gamblers equally. In divination, physical randomizers are fit for their purpose since the operator has no control over the outcome produced by the randomizer. Thus, if one believes that supernatural forces can control the outcome, their intervention explains the randomizer outcomes.

Section 7 examines a question that has occupied much of the recent literature. Why did ancient or even later thinkers prior to the seventeenth century fail to conceive the probability theory? I will suggest that the question presumes an inevitability in the emergence of the probability calculus. As a tonic, I will review just how extraordinary is the theory. Its concepts are abstruse, even today, and it is based on the remarkable idea that all chance systems, no matter their physical basis, conform to the one theory of chance. In place of the original question, I ask why we would expect such a remarkable theory ever to emerge. Section 8 offers brief conclusions.

2. Lotteries: local and global conceptions of chance

2.1. The operation of the lotteries

The lotteries considered here are of the simple type in which a fixed number of prizes are offered to purchasers of numbered tickets. In many nations, lotteries have been offered successively, then proscribed as illegal and then offered again as a convenient means of raising funds.[2] While there were earlier lotteries in Europe,[3] it is convenient to examine English lotteries, starting with that of 1569. It is recounted in some detail in Ashton (1893, Ch. 1). These lotteries became events of great public interest. The lottery drawing at Guildhall in London was of sufficient notoriety that there are many eighteenth-century engravings depicting the event. A famous one by Hogarth is dated to sometime after 1724. It is populated with Greek mythological figures. A more sober representation dated 1750 is shown in Fig. 1.[4]

Two large hollow wheels, perhaps a dozen feet in diameter, contain tickets. One has tickets corresponding to those sold. The other has blanks and prize names. Drawings from each then match ticket numbers with blanks or prizes. The caption of another 1739 engraving of the scene describes the procedure.[5] “Two Blue Coat Boys taking the Tickets out of the Wheels. Two men to take the Tickets from the Boys, and One names the No.$^{\text{brs}}$ of the Tickets, & the Other whether they are Blanks or Prizes …”

2.2. The local analysis

Starting in the seventeenth century, when local analyses of chance became standard, lotteries were singled out through local analyses as unfair wagers. In their Port-Royal Logic, Arnauld and Nicole (1662, p. 385; 1996, p. 272) give an early denunciation of lotteries:

> … there is an obvious injustice in the type of games called lotteries because, with the operator of the lottery usually taking a tenth for his share in advance, the whole group of players is duped in the same way as if someone made an equal wager, that is, where the likelihood of winning is as great as that of losing, of ten pistoles against nine. Now if this is disadvantageous to the whole group, it is also disadvantageous to each person in it, since from this it follows that the probability of loss exceeds the probability of winning by more than the advantage desired exceeds the disadvantage to which one is exposed, namely of losing what he has put in.

This denunciation became a routine illustration of the application of local chance analysis. It is included in the “lottery” entry in the Diderot/ d’Alembert *Encyclopédie* of 1765. The entry gives a calculation of what we would now call the expectation or expected value of a ticket in a fictional lottery (d’Alembert, 1765)[6]

> For example, let there be a lottery of 10 tickets at 20 livres per ticket, and let there be only a share of 150 livres, the expectation of each interested party is only 150/10 livre = 15 livre and his stake is 20 livre therefore he loses a quarter of his stake, and could sell his expectation for only 15 livre.

These analyses show that lotteries are not fit for the purposes of players if their goal is to secure fair wagers. Presumably the continuing flourishing of lotteries shows that this is not the purpose of players.

2.3. The global analysis

From the global perspective of those offering the lottery, chance does not enter into the analysis.[7] Rather, crown or government sponsored lotteries were touted as a palatable and popular way to raise funds as an alternative to taxation. There was no risk or chance. The operators knew that, if the revenue from tickets sold exceeds that of the prizes offered, their profit was assured. An anonymous eighteenth-century pamphleteer wrote (Anon, 1776, p. 5):

> Since then, the Mode of borrowing Money has always been a Bone of Contention in the House of Commons. It is the Business of the Chancellor of the Exchequer to find Ways and Means to raise the Supplies wanted, with as little Grievance to the Subject as possible. After every necessary Article of Life had been taxed, after even the Admission of the Light darted from Heaven into our Windows had been measured out to us, and every Thing we eat and drink heavily

2 A terse but revealing summary is given in Anon (1911).

3 For details, see Franklin (2015, p. 283).

4 Source: https://commons.wikimedia.org/wiki/File:Image_taken_from_page_1141_of_%27Old_and_New_London,_etc%27_(11186631394).jpg “Image extracted from page 1141 of volume 1 of Old and New London, Illustrated, by Walter Thornbury.”

5 Source: https://www.bridgemanimages.com/en/noartistknown/drawing-the-state-lottery-at-guildhall-1739-engraving/engraving/asset/239449.

6 Translated by Richard J. Pulskamp. Ann Arbor: Michigan Publishing, University of Michigan Library, 2009. http://hdl.handle.net/2027/spo.did2222.0001.119.

7 Matters would be otherwise in the case of a “Genoese lottery.” Players select numbers and win at prescribed odds if their numbers arise in a random drawing. To ensure a profit, the operators must antecedently adapt the odds offered to the chances of payouts. For a history, see Bellhouse (1991).

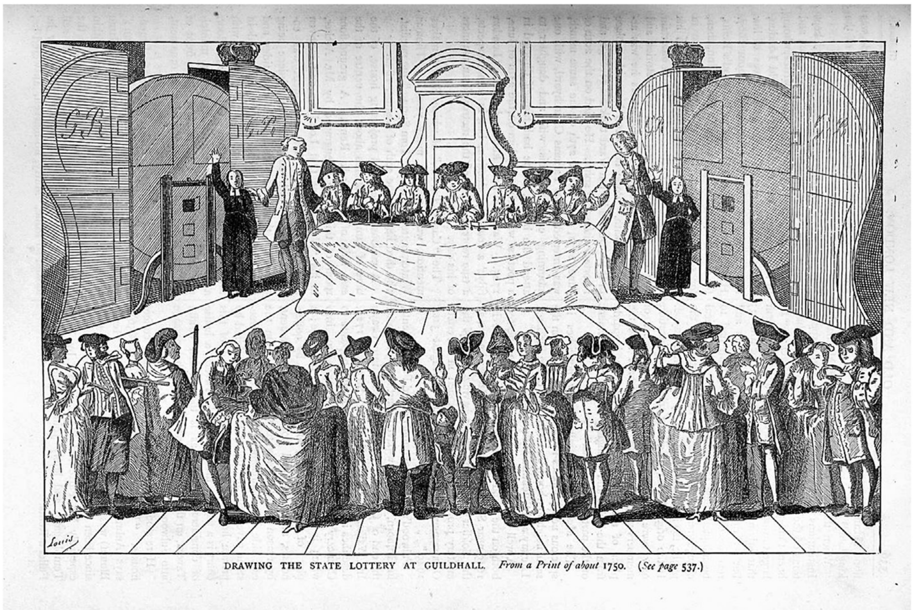

Fig. 1. Lottery drawing at Guildhall, London.

> laden with a Tax, what Mode of raising Money could be so equitable, so just or so pleasing to the Subject, as that by Way of Lottery?
>
> In raising Money by a Lottery, every Individual is at his free Option, whether or not he will contribute towards the Necessities of the State.

Correspondingly, the Diderot/d'Alembert *Encyclopedia* recounts (d'Alembert, 1765):

> Our French lotteries commonly have for their object to raise some funds destined to some pious works or to some need of the state; but lotteries are very frequent in England and in Holland, where one can make them only by permission of a magistrate.

This global analysis correctly reflects no doubt that lotteries are fit for their purpose of raising funds for the lottery operators.

2.4. *Independence of the global from the local analysis*

Global analysis has one advantage over local analysis: its results are independent of the local chances. To see it, purely as an artifice, we can imagine oddly constructed lotteries in which a simple, local analysis of the probabilities would fail. If the lottery tickets are numbered, in a long sequence of lotteries, the winning ticket in each is selected from a secret list. The list is so contrived that in many repeated plays, no stable frequencies arise for the winning tickets. At first, smaller numbers win; and then larger numbers; and so on. Since an objective notion of probability for the winning tickets entails that, very probably, frequencies will stabilize, the objective notion cannot be applied to the individual outcomes of the repeated lotteries. A generalized epistemic notion of probability might be applicable if no single epistemic probability measure over the ticket numbers, but a set of probability measures such as is employed in theories of imprecise probabilities.[8] In these and other comparably fanciful scenarios, a simple local analysis that employs just one probability measure would fail. No matter what, the global analysis would remain quite serviceable to the lottery operator. If the tally of fees of tickets sold exceeds the value of the prizes offered, a profit is assured.

[8] For a survey, see Bradley (2019).

3. Bookmakers

Lotteries are exceptional: at the global level, chance cannot turn against the lottery operator. If all the tickets are sold, there is no risk of loss to the operator. In more complicated examples, there may still be a chance of loss at the global level. One might imagine that this means that the global analyst must revert to local conceptions of chance in order to assess and mitigate the risk. The example of traditional bookmaking shows otherwise. Risk can be mitigated in other ways at the global level.

Traditionally, bookmakers—"bookies"—at a horse-racing track stand in front of a chalkboard or some other display. On them, they post the odds at which they will accept bets from punters over which horse will win or place in each race. A chalkboard enables the bookmakers to adjust the odds as they wish or need.

3.1. *The local analysis*

Punters use a local analysis. They assess the chance or probability of each horse's winning. They then seek bets at odds that are favorable, according to the punters' local assessment of the probabilities. Before learning the ways of bookmakers, it is easy to imagine that bookmakers ply their trade in much the same way. To survive in the business, they must be astute probabilists who can intuit the probabilities more accurately than punters. They can then offer punters odds favorable to the bookie. In the long run, offering odds slightly worse than fair would mean that the bookmaker's losses and gains would average out to leave the bookmaker with a steady profit.

3.2. *The global analysis*

While something along these lines can happen, it is not how bookmakers operate their books. For otherwise their profitability would depend on the accuracy of their probabilistic hunches, day in and day out. Instead, bookmakers seek to offer odds in a way that insulates them from the need for such judgments and makes a profit all but assured. It is difficult to find canonical statements in a reliable literature for how bookmakers set up their books. This no doubt reflects the shifting legal status historically of bookmaking between legality and illegality. Most writing on bookmaking reports legal prosecution of illegal bookmaking. However, I have located a few suitable sources.

The modern tradition of the bookmaker with a chalkboard is recent. The honorific of "first bookmaker" is traditionally bestowed upon Harry Ogden. He plied his trade on Newmarket Heath Racecourse in Suffolk, England, around 1795.[9] The mathematical principle behind the bookmaker's strategy was soon articulated in De Morgan's "Theory of Probabilities" (1847, p. 406, his emphasis):

> A systematic gambler, who has many transactions of the same kind, is a merchant whose speculations differ from those commonly called mercantile, only in the individual risks being somewhat greater. But even this is not always the case. To take the instance of a horse-race; suppose one person can find somebody to take any bet he offers, and that four horses, *A*, *B*, *C*, and *D*, are to start. He offers four bets against each, (he must take care not to bet in favour of any.) namely, *a* to *a'* against *A*, *b* to *b'* against *B*, *c* to *c'* against *C*, and *d* to *d'* against *D*. The following then are his contingencies:
>
> If A win, he gains $b'+c'+d-a$.
>
> If B win, he gains $a'+c'+d'-b$.
>
> If C win, he gains $a'+b'+d-c$.
>
> If D win, he gains $a'+b'+c'-d$.
>
> All he has to do then is to make the sum of every three consequents greater than the fourth antecedent. Suppose, for instance, he bets £10 against £ 10 on every horse; he must then win £20 whichever way the matter may go. Or he may bet 10 to 5 against *A*, 10 to 6 against *B*, 10 to 7 against *C*, and 10 to 8 against *D*. Then, if *A* win, he gains £11; if *B*, £10; if *C*, £9; if *D*, £8: that is, £3 worth of fair gambling, and £8 worth of—not always perhaps sheer roguery, but very often so, and generally something very like it. It is evidently a *fool-chase*, not a horse-race, which he is engaged in.

Proctor's (1887) popular admonition on gambling devoted a chapter to the bookmakers' practices. It was derived from direct contact with bookmakers during a visit to Australian tracks. He professed Victorian disapproval (p. 104): "I regard betting as essentially immoral so soon as its true nature is recognized. … money has passed from one person to another without any 'work done' by which society is benefited …." His overt disapproval must be tempered by the obvious relish with which he wrote about the immorality.

Proctor (pp. 119–23) gave more details of what he called the "rules for success" of a bookmaker. A simple rule for balance is that the totality of odds offered, when reduced to probabilities, should sum to slightly more than unity.[10] The more interesting rule is (p. 120):

He should proportion his wagers so that the sum of what he lays against a horse, and what he is backed for, may amount to about the same for each horse.

Following this rule protects the bookmaker from accepting too many bets on just one horse, so that if that horse wins, the bookmaker makes a sure loss. Proctor's example of a well-balanced book for horses *A* to *K* has the bookmaker accepting amounts in total (p. 122)[11]

> £800 to £300 against *A*
>
> £900 to £200 " *B*
>
> £950 to £150 " *C*
>
> £980 to £120 " *D*
>
> £980 to £120 " *E*
>
> £1,000 to £100 " *F*
>
> £1,000 to £100 " *G*
>
> £1,040 to £60 " *H*
>
> £1,050 to £50 " *K*

The total stakes the bookmaker receives for bets on each horse sums to £1200. The bookmaker will make a net profit of £100 no matter which horse wins. (e.g. if horse *A* wins, the bookmaker pays the winning punters £800 + £300 = £1,100, since the original stake is returned.)

The general principle remains to the present. A recent practical reckoner gives a synopsis of horse racing conventions and useful charts for computing many of the figures needed. It concludes with a summary of bookmaking practices (Anon, 2007, p. 185, emphasis in original):

> The Fundamentals of Bookmaking.
> *It is the aim of the bookmaker at all times to balance his book so that whichever horse wins he will show a profit. Needless to say, this is not always possible, but at least it is the ideal to be aimed at.*

The key is that judgments of the probabilities of individual horses winning are incidental to the bookmaker's practice and best avoided. Such judgments are made by the punters. They take the risks of errors of judgment. The bookmaker takes no such risks. The bookmaker sets the odds in response to the distribution of bets placed by the punters. If a horse attracts many bets, it is designated a "favorite" and the bookmaker offers only shorter odds. By making continual adjustments to the odds offered, the bookmaker strives to maintain an overall portfolio of bets that gives a profit no matter which horse wins. If the bookmaker finds that the bets accepted have thrown his portfolio out of balance so that a loss will result if a newly favored horse wins, the bookmaker may restore the balance by "laying off." That is, the bookmaker would go to other bookmakers and place bets *on* the newly favored horse.

The activity is one of global analysis. For the bookie cannot offer odds comfortably, until the bookie has access globally to the bets punters are willing to make. If the bookmaker can adhere to the fundamental principle of keeping the book in balance globally, then the combined system of horse racing and betting is fit for the bookie's purpose of an assured profit.

3.3. The global approach mechanized

Since bookmaking has offered too many opportunities for dishonest practices that are unfair to punters, racecourse betting has been replaced in many venues by a "parimutuel" or totalizer system. It mechanizes the bookmaker's practices, now made transparent to punters. All the stakes are put in a pool. A percentage is withdrawn to cover the costs of operations and profit. The remaining stakes are divided among successful punters according to the size of their bets. Prior to the race, a totalizer machine mechanically computes, displays and continually updates the effective odds of a bet on each horse. These odds are computed from the bets already placed, just as would a bookmaker, but they are now offered as a service and an enticement to punters. A parimutuel system has eliminated the bookmaker in favor of something similar to a lottery. Aggregated contributions by players are redistributed to the few selected by a randomizer, the outcome of a race.

3.4. Independence of the global from the local analysis

Once again, the global analysis can be seen to be independent of whatever chance properties may prevail locally. As an artifice, imagine

[9] For more details on the dating, see Munting (1996, pp. 89–90).

[10] In a ten-horse race, the odds of 9 to 1 on each would correspond to probabilities 1/10 on each, which sums to unity. A book balanced to turn a slight profit would then offer odds of, say, 8 to 1 on each. The corresponding probabilities sum to 10 x (1/9) = 10/9 > 1.

[11] Proctor's table as displayed on his p.122 differs in giving amounts £900, £1,000, £1,050, £1,080, £1,080, £1,100, £1,100, £1,140, £1150 in the first column. These are the least favorable, but still admissible amounts the bookmaker can accept. The analysis provided by Proctor's text, however, presumes the figures in the first column as given above in the main text.

bookmakers at a track in which, unknown to them, nefarious forces contrive the outcomes so that ordinary sorts of probabilistic analysis fail. A strong horse with an excellent record of wins may sudden be made to perform poorly. It is not just that past frequencies cannot be projected. It is that they are contrived to mislead. This would be disastrous for the individual punter. However, if a bookmaker keeps a balanced book, the overall system would remain fit for the bookmaker's purposes.

4. Ancient physical randomizers

The use of physical randomizers permeates ancient civilizations. Here are three uses of them.

4.1. Games and gambling

We learn from extensive histories of early chance systems that gaming and gambling using physical randomizers was pervasive in ancient cultures. See for example David (1955; 1962, Ch. 1) and Franklin (2015, Ch. 11). Some were rudimentary and merely involved the casting of nuts[12] or flat objects like pieces of horn or butter beans.[13] Others were better crafted. Physical randomizers recovered in archeological investigations include at least some that we would even now consider serviceable. They include cubical dice and dice shaped from other regular solids, such as tetrahedral dice used in ancient Egyptian games, and even ancient, twenty-sided, icosahedral dice. Here we will consider what both David (1962, p. 6) and Franklin (2015, p. 290) find to be the principal or most common randomizer used in ancient games of chance: astragali (Greek), tali (Latin) or, as they are commonly known to children today, knucklebones. They are the heel bones, typically, of hooved animals, such as deer and are irregularly shaped, six-sided objects. Two of the ends are rounded and cannot arise in a cast. Two opposing flatter sides, numbered 3 and 4, arise more easily; and two opposing narrower sides, numbered 1 and 6, arise less easily. David (1955, p. 3) estimates that the flatter sides arise each roughly with probability 4/10 and the narrower sides each with probability 1/10. Since each astragalus is unique in its shape, precise probabilistic analysis is precluded for astragali in general.[14]

The ancients gambled with energy and passion. Steinmetz (1870, Ch. 4) surveys the gambling excesses of Roman emperors. While the games are often mentioned in classical literature, it has been harder to discern precise rules of play with astragali. A common game, described in Toner, 1995, (Ch. 8), resembles the modern Channukah game of dreidel (which may descend from Greek or Roman versions of comparable games). Gamblers contribute or draw from a pot of money. Which they do is decided by a cast of four astragali or tali. The most favorable cast is the "Aphrodite" or "Venus," in which all four astragali show different numbers. The lucky gambler who throws it takes the entire pot. The least favorable casts are one's, called "dogs." If they are thrown, the gambler must contribute to the pot.

[12] As reported in Lanciani, 1892, pp. 100–101).

[13] As reported in Franklin (2015, p. 290).

[14] An enduring puzzle in cubical dice design is why the pips were commonly arranged with opposing sides summing to seven. Cardano (1663, p. 233) suggested: "This is done with the idea of making it easier to detect falsification, if anyone should make a false die, by duplicating the one and leaving out some other number." It is an implausible explanation. If the duplicated one replaces the opposing six, then the duplication would be invisible from any fixed vantage point. This arrangement of pips on astragali make sense. Then the middle values of 3 and 4 are cast more easily and the extremes of 1 and 6 less so. Might this familiar arrangement be carried over to newer cubical dice even though its rationale is lost? An observation by Eerkens and de Voogt (2017, p. 171) contradicts this suggestion. They found that irregular, flattened dice from before 650BCE in their sample carried pip counts of 1 and 6 on opposite flattened sides.

4.2. Lots and lotteries

Making decisions by casting lots was routine in the ancient world. It is mentioned frequently in the Bible, mostly in the Old Testament. The most common application is the division of inheritances and conquests. The most extended example is in nine chapters of Joshua (Ch. 13–21). They chronicle in detail how the conquered land of Canaan is to be divided among the people of the tribes. While copious details of the outcomes are provided, the text gives no details about the actual casting of lots. The repeated verbiage is "divided by lot" or some simple variant. This omission leaves obscure the precise procedure followed. Minimally we can assume that some physical randomization was involved.

Psalm 22 reflects how commonly decisions were taken by lot. The psalmist laments how tormentors divided the psalmist's clothing by "cast[ing] lots upon my vesture."[15] This lament anticipates the New Testament reports in the gospels of Matthew, Mark, Luke and John: after Jesus was crucified, "they parted his garments, casting lots upon them."[16]

Sortition is the use of lotteries or comparable randomizing devices to select appointments to positions of political authority. Aristotle's[17] *The Athenian Constitution* shows the practice had been integrated thoroughly into the governance of Athens. The word "lot" appears roughly 100 times in the Loeb translation (Rackham, 1935). The narrative is both historical, describing past practices, and prescriptive. In both contexts, the drawing of lots was used to select almost every imaginable position. In the prescriptive part, membership in the ruling Council was to be decided by lot; and the office of President rotated among the tribes in an order determined by lot (Rackham, 1935, p. 123). Selection by lot continued for appointment after appointment. Lots decided who would be the port superintendents, the prison wardens, the judges, the Restorer of Temples, and even the Receivers who are authorized to wipe clean records of debts on tablets once paid (p. 133). Lots were to be drawn to decide who would have an audience with the Council (p. 93). The drawing of lots was so pervasive that the Constitution had to specify when they were *not* to be used: when Generals seek an audience with the Council concerning matters of war. (p. 93)

No procedure is specified for the drawing of lots in almost all the simple cases. That changes at the end of the extant document in the more complicated matter of assigning juries to different courts. Ch. LXIV (pp. 173–175) describes a complicated lottery system for determining who serves in which courts. The description is elaborate and includes lettered acorns drawn from urns; wooden tickets with each candidate's name, his father's, his tribe and a letter; multiple wooden boxes from which they can be drawn; and white and black dice. Later reconstructions of the procedure connect it with a kleroterion. It is a flat, upright stone with multiple columns of slots into which citizens' identifying tickets could be placed.[18]

4.3. Divination

The practice of divination occurred in almost all ancient cultures and permeated ancient life. The cultural breadth and range of methods is reflected in Engels and Nice's (2021) survey. It covers Mesopotamia, Egypt, Iran, Greece, Etruria, Republican and Imperial Rome; and is not exhaustive.[19] "Cleromancy" or "sortilege" is a form of divination in which the outcome of the operation of a physical randomizer, such a drawn lots or cast astragali or dice, is interpreted prophetically. Its

[15] King James Version, 1611, Psalms, 22:18.

[16] King James Version, 1611, Mark, 15:24.

[17] Thorley (2004, p. 92) reports that the text was probably written by one of Aristotle's pupils.

[18] For a more detailed account of the use of the kleroterion, see http://www.agathe.gr/democracy/the_jury.html.

[19] See Heiduk (2021) for a survey covering the Medieval Western Christian World.

practice was widespread in antiquity and has attracted an extensive secondary literature (see for example Luijendijk & Klingshirn, 2019).

Duval (2015) provides a concrete example of precisely how physical randomizers were used for divination in the ancient Greek world. She reports 21 sites in Asia Minor from the second century CE, all of which used the same text to divine the meaning of the cast of five astragali. The texts were publicly available at the sites as inscriptions on stone monoliths. They matched each of the 56 unordered, possible casts of five astragali with their interpretations. Here are two examples (p. 129):

> 1,1,1,1,3 [sum to] 7 Athena Areia
>
> If four Chians and one three are cast, the god signals:
>
> By avoiding enmity and animosity, you will reach your prize;
>
> you will arrive and the blue-eyed goddess Athena will save you.
>
> The activity that you have in mind will turn out as you wish it.
>
> 1,1,1,1,4 [sum to] 8 Moirai
>
> If one four and four Chians in a row are cast:
>
> Don't do the business that you are engaged in; it will not turn out well.
>
> It will be difficult and impossible around someone who tires himself out.
>
> But if you go abroad for some time, no harm will come from it.

Cleromancy was practiced widely in other cultures. While divination was discouraged in the bible, there is evidence of it and specifically of sortilege. The clearest case concerns the fate of Jonah. Lots were cast to discern who on the storm-tossed boat was the cause of the danger and should be cast into the waves. (Jonah 1:7)[20]

> And they said every one to his fellow, Come, and let us cast lots, that we may know for whose cause this evil is upon us. So they cast lots, and the lot fell upon Jonah.

Tacitus (Townshend, 1894, pp. 62–63) in his *Germania* reported how cleromancy was practiced in the German tribes. Marked twigs were cast onto a white robe and then three were drawn, one at a time, and interpreted.

The urge to attach prophetic significance to the outcome of physical randomizers is widespread, enduring and insatiable. A nineteenth compendium (Smedley, Taylor, & Thompson, 1855, Part V) found historical records supporting nearly 30 different forms of "… mancies." The common element of many was the finding of significance in outcomes produced by simple physical processes that otherwise are otherwise random. "Geomancy" interpreted the patterns formed by cast pebbles. Hydromancy interpreted patterns in disturbed water. Alectromancy interpreted the lettered grains selected by a prepared cock through its pecking. Dactylomancy interpreted the direction of swinging of a ring suspended by a thread.[21] Axinomancy interpreted the motion of a balanced ax. These accompany more familiar examples, such as sortilege, which is defined as divination by lots, and cleromancy, which employs thrown objects such as beans and dice.

5. The ancient comments

5.1. Games and gambling

Given the prevalence of the use of physical randomizers in antiquity, one would expect that there is at least a small ancient literature that explains why they are fit for their purposes. The literature sought here is not about notions of chance and determinism, fortune and fate in general. Those topics have attracted much ancient attention and an enormous modern historical literature. What is sought are accounts *specifically addressing the behavior of physical randomizers* and their fitness for their purposes. The present literature has identified only a few remarks that show some understanding of the chance behavior of physical randomizers; and these few appear regularly of necessity in the historical literature, since they are all we have. They fall short of what is needed.

A few remarks show an ancient understanding that some extreme combinations of outcomes of astragali casts are rare and unlikely. Aristotle wrote in *de Caelo* (Book II, 292a30; Barnes, 1984, p. 481):

> To succeed often or in many things is difficult. For instance, to throw ten thousand Chians [ones] with the astragali[22] would be impossible, but to throw one or two is comparatively easy. In action, again, when A has to be done to get B, B to get C, and C to get D, one step or two present little difficulty, but as the series extends the difficulty grows.

In a similar vein, Cicero remarked (Falconer, 1923, pp. 249–51):

> 'Mere accidents,' you say. Now, really is that so? Can anything be an 'accident' which bears upon itself every mark of truth? Four tali[23] are cast and a Venus[24] throw results—that is chance; but do you think it would be chance, too, if on one hundred casts you made one hundred Venus throws.

Corresponding to Cicero's observation that a single Venus cast is unremarkable, Aristotle also recognized that ordinary outcomes are to be expected by chance. In repudiating the prophetic power of dreams in his "On Divination in Sleep," Aristotle noted (463b18-22, Barnes, 1984, p. 737):

> … they just chance to have visions resembling objective facts, their luck in these matters being merely like that of persons who play at 'odd and even.'[25] For the principle which is expressed in the gambler's maxim: "If you make many throws your luck must change," holds good in their case also.

These passages from Aristotle and Cicero were written three centuries apart, two in the Greek literature and one in the Roman literature. Their intent is not to instruct us about the chance behavior of astragali or tali. Rather they are drawing on a presumption that readers, centuries and lands apart, are familiar with these chance behaviors. They use that familiarity to make points in other areas. While—frustratingly—these passages do not give us a clear statement of the ancient view of the chance properties of astragali and tali, we do see that there was an assumption of widespread agreement on at least some aspects of them. This suggests an explanation for the lacuna in the ancient literature. When something is in the realm of "what everyone knows," there is little incentive to write about it.

5.2. Cleromancy and sortilege

The literature on cleromancy and sortilege, however, is different from that of simple games and gambling, but not in a helpful way. The extant texts argue that physical randomizers are *not fit* for divination. Foremost among these is Cicero's *On Divination.* It is a sustained, devastating refutation and exposé of ancient divination.

[20] King James Version, 1611.

[21] This one I was shown in my youth as a means of ascertaining the gender of the child of an expectant mother.

[22] The word "dice" in the translation has been replaced by "astragali" since that word was used in the Greek text. For present purposes, here and below, it is important to correct the frequent mistranslations of astragali and tali as dice. The former do not have equal chance outcomes readily amenable to local analysis, where the latter (dice) do, if they are perfectly regular.

[23] The correct "tali" replaces the translation's incorrect "dice."

[24] The highest value cast with all four faces different.

[25] Here I follow Franklin (2015, p. 290) and replace the word "dice" in the translation by "odd and even."

A few extracts give a sense of its power. “Of what advantage to me is divination if everything is ruled by Fate?” the text asks (Falconer, 1923, p. 392). Suppose that disasters, such as the accidental death of a friend or the loss a fleet, could be predicted reliably. Then knowing the prediction would be useless, since the disasters would be inevitable. Yet if the prediction enabled the disaster to be averted, then the prediction was wrong.

The whole idea of sortilege is dismissed as foolish superstition or fraud (Falconer, 1923, p. 467):

> And pray what is the need, do you think, to talk about the casting of lots? It is much like playing at mora, dice, or knuckle-bones, in which recklessness and luck prevail rather than reflection and judgement. The whole scheme of divination by lots was fraudulently contrived from mercenary motives, or as a means of encouraging superstition and error.

After some analysis (p. 469):

> This sort of divining, however, has now been discarded by general usage. The beauty and age of the temple still preserve the name of the lots of Praeneste—that is, among the common people, for no magistrate and no man of any reputation ever consults them; but in all other places lots have gone entirely out of use.

Klingshirn (2019) contains compiles the writings of other ancient and later authors who similarly disparage sortilege.

6. The global analysis

6.1. Equality

The ancients did find physical randomizers to be fit for their purposes in games, gambling, sortition and cleromancy. Without direct accounts by them of why they found physical randomizers to be fit for these purposes, we can only make plausible suppositions. My conjecture here is that their analysis was global and that, for gambling and drawing lots, the fitness for purpose followed from a property of their chance systems at this global level[26]

> *Equality*. Each participant has identical opportunities and is governed by identical rules.

This principle implements and generalizes Kidd's (2020) observation that ancient gamblers faced chance communally and not individually (pp. 2–3):

> The gambling that took place in antiquity tended to be played at a communal risk, where the wager involved was a group-wager agreed upon by everyone ahead of time. The outcomes, of course, were individualized—some gamblers won and others lost, sometimes large sums of money—but the risk itself was shared equally before the game commenced and the actual dicing began.

In standard gambling play, each player casts the astragali or tali and, according to the outcome, contributes to the pot or takes from it under the same rules. Each gambler is an equal participant; none is favored. The same equality applies in the use of lots and sortition.

That the chances are treated communally is, of course, not enough. The communal engagement must be such that no participant is favored; and equality must be implemented in the governing rules. A gambling game in which one player only is allowed multiple casts of the astragali would be communal, but unfair and fail *Equality*. A lot drawing in which one participant only is allowed multiple tickets in the urn would be similarly unfair.

That *Equality* is implemented at the global level explains how it is possible for ancient gamblers to be comfortable using irregular randomizers like astragali. An assurance of equality does not require a local understanding of the probabilities of individual casts. Equality is assured by the overall rules of their games, which must treat each gambler equally. It does not matter to a gambler if some sides arise more easily, as long as that ease does not provide an advantage to other gamblers; and it does not if all the gamblers play by the same rules.

The closest statement of this idea of equality is given later by Cardano in work published in the seventeenth century. He articulated the “Fundamental Principle of Gambling” (Cardano, 1663, p. 189):

> The most fundamental principle of all in gambling is simply equal conditions, e.g. of opponents, of bystanders, of money, of situation, of the dice box, and of the die itself. To the extent to which you depart from that equality, if it is in your opponent’s favor, you are a fool, and if in your own, you are unjust.

The principle is in one aspect more expansive than *Equality* above, since it demands equality in many further conditions; and it is less expansive since it does not explicitly call for equality in the rules governing the play.

The continued popularity of irregularly shaped astragali or tali indicates that the ancients were content with the global level of analysis. Local analysis is difficult. The chances of the different faces are unequal and they will vary from astragali to astragali as their shapes vary. Adequately treating these differences lies beyond the recorded methods of antiquity. A comfort with irregular shaped physical randomizers is also supported by archeologically recovered ancient dice. While there were many cubical dice, there were also irregular dice that deviated markedly from cubical form. Artioli, Nociti, & Angelini, 2011 examined the physical characteristics of Etrurian dice from the eighth to the third centuries BCE. In their tabulation of 91 samples, 74 are listed as “cube” in shape and 17 as “parallelepiped.” Ignatiadou (2019, p. 147 and Fg. 4) reports a rare rhomboid die from the late 4th to early 3rd century BCE. Its faces were of different sizes and included parallelogram shapes with both acute and obtuse angles.

While fabricating regular cubical dice was within the capacities of ancient crafts, a preference for irregular astragali or tali remained. A curiosity in the numbering of astragali is that their broad faces are numbered 3 and 4 and their narrower faces 1 and 6. Why not number them 1, 2, 3 and 4? The obvious explanation is that the numbering is copied from the six-sided cubical dice already in use. If this explanation is correct, we affirm that ancient players were untroubled by the transition from regular cubes to irregular astragali. They would be unperturbed if their analysis was global. Tschen-Emmons (2014, p. 70) reported that the Romans went to the trouble of manufacturing artificial tali. Once again, if their analysis was global, these Romans were just continuing an unproblematic tradition. If their analysis was local, this fabrication is inexplicable. They would be visiting unnecessary, problematic complications on their analysis, where the same efforts could have produced cubical dice, amenable to local analysis.

The archeology of dice in the first and second millennium CE indicates a transition from global to local analysis. Eerkens and de Voogt (2017) collected dice that were dated to 0–1900 CE from museums and archeological depots in the Netherlands. They judged a die to be noticeably asymmetric if the maximum dimension exceeded the minimum by 5%. Nearly 90% of the dice dated to 0-650 BCE were asymmetric by this standard. Less than 40% were asymmetric for dice dated after 1450 BCE. This transition suggests a growing concern for dice that would return each face with equal chances.

If participants are assured of *Equality*, then gamblers know that the game is fair in the sense that no gambler has any advantage; and participants in a lot drawing know that a comparable fairness applies to the

[26] We might now call this condition “symmetry.” I use “equality” instead, since it is closer to the terms used historically. The modern usage of “symmetry,” I suspect, may not have been established fully until the application of group theory to quantum mechanics in the first part of the twentieth century. For a survey, see Hon and Goldstein (2008).

lot's selections. *Equality* establishes the fitness-for-purpose of physical randomizers in these two cases.

6.2. Lack of control

Equality is not obviously a condition required for the fitness-for-purpose in cleromancy. The practice does not apply to a collection of participants, each expecting equal opportunities. It involves an oracle and a person who consults it. It is harder to determine what suitable conditions might be. The idea that cleromancy works at all was already subject to telling dispute in antiquity. If these skeptics are right, then no conditions can make a physical randomizer fit for its purpose in cleromancy. We can ask a slightly different question. If contrary to Cicero and Aristotle's admonitions, one accepts that something like divination works, what is it about the chance systems of cleromancy that make them fit for their purpose? There is a natural candidate:

> *Lack of control*: the operator of the physical randomizer has no control on the outcome.

Someone consulting an oracle must be confident that the forecast derives from a source other than the operator of the physical randomizer. While it might be welcome to have the oracle cast a favorable throw—perhaps a Venus—its prophetic significance is nullified if one suspects that the operator can cast Venuses at will. That the operator does not know the randomizer outcome in advance and may be surprised by it is part of the theatrics of cleromancy. It is part of modern day I Ching forecasting and tarot card reading.

Gataker's (1619, pp. 3–4) historical and theological treatise on lots and lotteries[27] collected quotes from "good Authors,"[28] rendered in his italics below. They support the conclusion that the distinctive feature of lots and the like is that their outcome is outside our control. Gataker first considered the entrapment of a thieving servant by tempting the servant with a bait of money. It is contrasted with a method of detection that employs a random process:

> Whereas the matter of a Lot is euer some euent meerely casuall; as if a man to try whether his seruant be a theefe or no, shall put a scroll with his name in it, togither with others rolled vp seuerally into water, to see which will vnfold first, and thereby to determine and iudge of the party suspected whether he be guilty or guiltlesse of that crime. To which purpose tend those sayings of good Authors, that *To vse Loterie is to put a thing from skill and counsell to temeritie and casualty*. that *A Lot is the child of chance*. that *The issue of Lots is not in mans power, but is such as casualty casteth on vs*. that *In Lotery there is no certainty*. that *Lots are not carried by reason and iudgement, nor by counsell and aduice*: but *Chance and casualtie striketh the cheife stroke in them*; if wee respect secondary causes.

In so far as one accepts that the gods or other sources can communicate forecasts by manipulating physical randomizers, the requirement of *Lack of control* suffices to establish the fitness for purpose of a physical randomizer for cleromancy, while that fitness does not require that every randomizer outcome is such a communication.

7. Why didn't they ….?

7.1. The question as usually treated

The question prominent in the existing literature is why probability theory did not emerge earlier. The first forms of the theory employed nothing abstruse mathematically. It required counting, some simple combinatorics and the taking of ratios–all well within the reach of ancient mathematicians. Yet it took millennia for the theory to emerge. This was so even after the transition from a global to a local understanding of chance was underway. Its slow pace is evident artifactually. After antiquity, perfectly cubical dice began to replace irregular dice and astragali. Their perfect form indicated that their users understood the starting point of local analysis: that the chance of each side of a cubical die is the same.

Many answers have been proposed. Kendall (1956, p. 10) lists the absence of combinatorics or a notion of chance events, superstition or moral or religious barriers. David (1962, Ch. 3) weighs many possibilities in a chapter's length discussion. Irregularly shaped ancient randomizers, for one, precluded discovery of equal chance outcomes and even the stability of statistical ratios. Sambursky (1956) conjectures multiple factors for the omission: the lack of combinatorics (p. 44); belief in good or bad luck (p. 45); opposition to regularity in the sublunar world of Aristotle (p. 47); and the absence of systematic experimentation (p. 47). Franklin (2015, pp. 330–40) gives a similarly extended analysis, some of whose points are noted in Section 7.2 below. He notes that die casting is unlike real, chancy situations (p. 335). Hacking (2006, Ch. 1) rejects several possible answers: "a necessitarian view of the world, piety, lack of a place system of numeration and of economic incentive." (p. 8) In their place, Hacking offers the theme of his book, the dual nature of probability, as a necessary precondition for probability. The dual nature merges stable frequencies and degrees of belief. The precondition is excessive. It precludes by fiat probability as a measure of purely physical chances, independent of human thoughts and beliefs. Kidd's (2020) analysis is closest to that of this paper: ancient gambling games could be analyzed globally and thus had no need of the local analysis of probability theory.

7.2. Why did it ever … ?

Intriguing as this last question is, we have been induced to focus on the wrong question. A hidden assumption makes the slow emergence appear puzzling. It is that there is something natural and even inevitable about probability theory. It is treated akin to old world sailors, who must inevitably find the new world if only they venture to sail far enough west. What took them so long?

This assumption is mistaken. The unnaturalness of a general theory of chance means its discovery is not inevitable. Its discovery is remarkable. Many obstacles must be passed before such a theory can be secured. Here are a few:

1. Chance marks the boundary of where regular theories can be deployed.

Is that not what we mean by chance? Once we enter its realm, regularity is lost and, with it, any hope of a regular theory. Cicero makes the point repeatedly in his repudiation of divination (Falconer, 1923, p. 387):

> Can there, then, be any foreknowledge of things for whose happening no reason exists? For we do not apply the words "chance," "luck," "accident," or "casualty" except to an event which has so occurred or happened that it either might not have occurred at all, or might have occurred in any other way. How, then, is it possible to foresee and to predict an event that happens at random, as the result of blind accident, or of unstable chance?

Cicero's denunciation becomes more colorful (pp. 389–91)[29]

[27] Bellhouse (1988, p. 67) reports that the terms "lots" and "lotteries" had a broader meaning in Gataker's literature than the modern ones. They include the general use of randomizers such as dice and cards.

[28] A sidebar in Gataker's text includes terse citations to these authors, naming Cicero, Euripides and Ambrose, and gives the original Latin texts.

[29] These two passages do not mention tali or dice explicitly. Their casting is surely included in Cicero's conception of chance. He wrote elsewhere (Falconer, 1923, p. 507): "Nothing is so uncertain as a cast of tali …" where "tali" replaces the translator's incorrect "dice."

> Surely nothing is so at variance with reason and stability as chance. Hence it seems to me that it is not in the power even of God himself to know what event is going to happen accidentally and by chance. For if He knows, then the event is certain to happen; but if it is certain to happen, chance does not exist. And yet chance does exist, therefore there is no foreknowledge of things that happen by chance.

Franklin (2015, p. 335) finds remarks of Aristotle that would seem to preclude any theory of regularity concerning chance, but without specific mention of physical randomizers like astragali.

This is not our modern view. Results produced by probabilistic processes are replete with hidden regularities. Modern analysis is adept at using them to detect fraudulently fabricated data. That this is possible is astonishing even to us today and surely beyond the expectations of the ancients. The global approach accepts the inscrutability of chance and still manages to arrive at serviceable results. It is natural to think that this global circumscription does as much as any reliable analysis could do.

2. Each physical randomizer is a distinct physical device and, in principle, requires its own chance theory.

If we are lucky, we might find a physical theory for the tumbling of astragali. If we are lucky again, may find another theory for the mixing of lots in an urn. It would be extraordinary if they were both the same theory and even more extraordinary if that one theory could cover all imaginable physical randomizers. Yet just that is what probability theory proposes to do.

3. While the mathematics may be simple, the concepts of probability theory are not.

To construct even a simple theory requires many smaller technical problems to be solved. Here are two.

3a. We must have the ability to count combinations correctly.

If we cast two dice, casting two sixes is a single case. Casting a six and a five is two cases. We all learn to chant in our probabilistic schooling: "Six on the first die and five on the second; and vice versa." But why not "Six on one of the dice and five on the other"? Why is it not one case?[30] The result is not a matter of logic or mathematics, but of the physics of die casts: the outcomes on each die are physically independent. The physics can be otherwise. In bosonic statistics in quantum theory, this independence fails when we count photons. A new statistics of case counting is required. Franklin (2015, p. 331) noted:

> To obtain correct answers, one must have clear either the distinction between permutations and combinations (or partitions and ways of falling) or the concept of independence of events, at least sufficiently to know when probabilities should be multiplied. These notions are very difficult.

3b. We must formulate correctly the delicate connection between frequencies and chances.

As Cardano (1663, p. 192) pointed out, cast a die six times and we almost certainly will not have what the equation of chances and frequencies seems to demand: one of each face. Securing a serviceable account of the connection between chance and frequency was difficult and late in coming. Norton (manuscript) argues that it had not been made securely even in the seventeenth century. Prominent analyses of chance, such as that of Huygens (1657), proceeded without it. The connection came with Jacob Bernoulli's (1713, Part IV) version of the law of large numbers in the early eighteenth century. Probabilities and frequencies are even then not directly connected. Probabilities are connected with the probability of frequencies. We accept the circularity, but it is foundationally irksome.

4. While we may treat probability as an unproblematic concept, the later history shows otherwise.

Franklin's (2015, p. xx) synoptic diagnosis is:

> The main part of the answer lies in appreciating just how difficult it is to make concepts precise, especially when mathematical precision is asked for in an area that seems at first glance to be imprecise by nature.

What *is* probability? Later foundational analyses of probability have failed to answer univocally. Is it some form of a limiting frequency? A measure akin to an area? A propensity of a chancy system? A logical relation among propositions? Is it a credence constrained by principles of rational decision making? Should its measures be countably additive or merely finitely additive? Is it one of these in one case and a different one in another? Or is it everywhere some combination of these? There is no universally accepted answer to these questions. Partisans still fiercely defend their own favored interpretation.

When all these obstacles are considered, the pressing question ceases to be "why did probability theory not appear earlier?" It becomes "why did probability theory ever emerge at all?" A major part of the answer must be that it emerged from the determined efforts of a small number of seventeenth and eighteenth century theorists. What resulted was not a perfected theory but a fertile program of research that continues to develop today.

8. Conclusion

The probability calculus has provided us with an analytic tool of enormous power and scope. It is now tempting to imagine that it provides the only way to analyze chance systems. Global analysis provides an alternative. It is mostly weaker than probabilistic analysis. It is often the poorer choice, but not always. If we run a lottery, a profit is assured if the revenue from ticket sales exceeds the value of the prizes. If we make book at the racetrack, we are secure against loss if we maintain a balanced book. Astragali are irregular physical randomizers for which local analysis is difficult. The ancients who favored their use in gambling needed only to ensure that the rules of play treated all equally. Then global analysis could assure them that the game is fair. We can ask why they did not devise a local probability theory. The simple answer is that their games did not require it.

[30] The standard counting was already identified in the thirteen century de Vetula (See Bellhouse, 2000, for details.). That hundreds of years were still to pass before probability theory emerged in a more modern form shows that combinatorics is not enough.

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