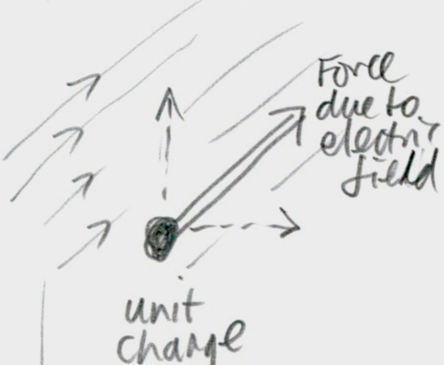
Mininum Maxwell

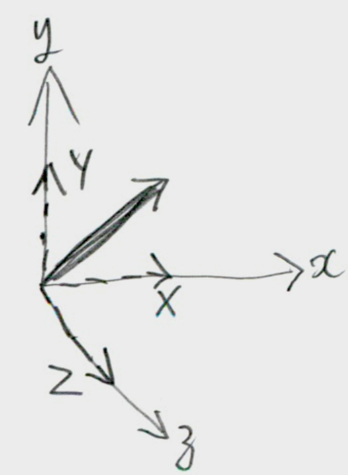
The mininum of Maxwell's equations needed for Einstein's 1905 special relativity paper

Charge free empty space is filled with two fields

Electric field strength

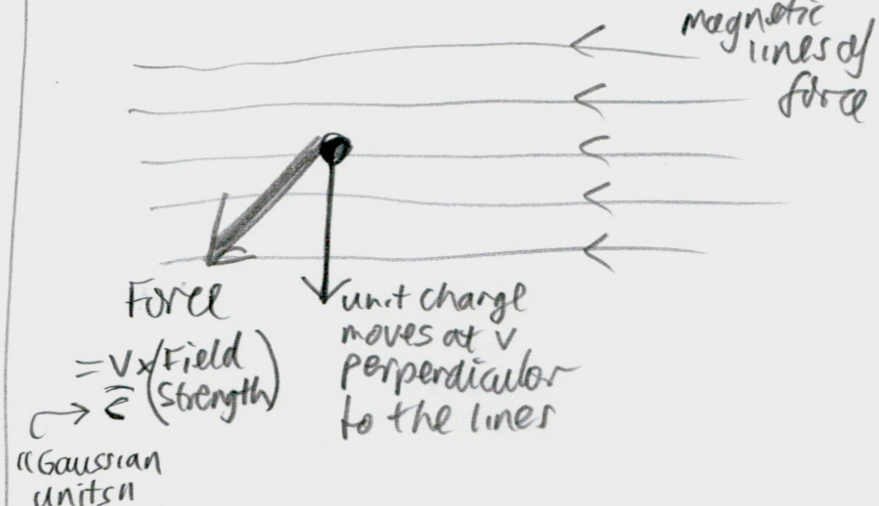


components of fieldstrength

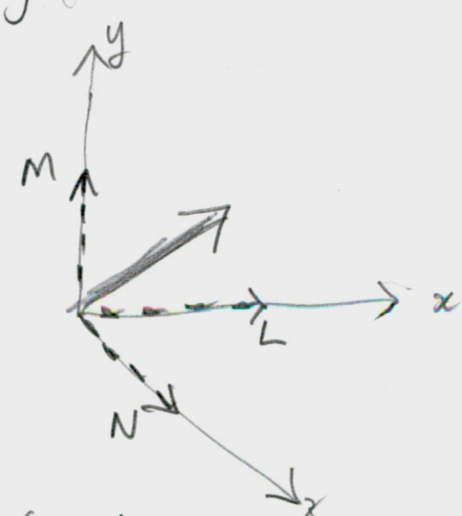


(modern $E = (X, Y, Z)$)

magnetic field strength



components of field strength



(modern $H = (L, M, N)$)

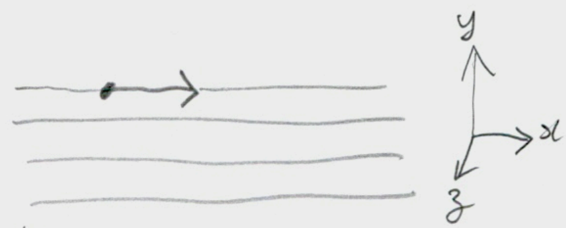
Maxwell's four (vector) equations $\longrightarrow$ How the fields
$X(x,y,z)$, $Y(x,y,z)$, $Z(x,y,z)$
$L(x,y,z)$, $M(x,y,z)$, $N(x,y,z)$
are spread over space x, y, z
& how they vary with time t

source free case
No charges

I $$\boxed{\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0}$$

says electric field lines of force are conserved

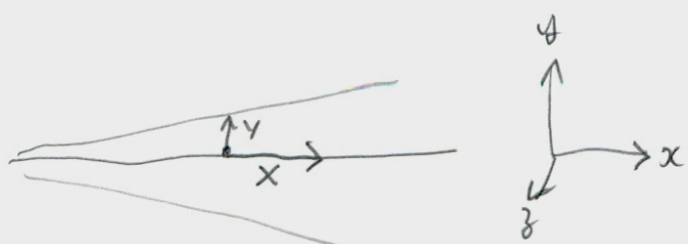
Motivation Homogeneous case



Lines point in one direction } x-direction

$$\left.\begin{array}{l} Y = Z = 0 \\ \therefore \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0 \end{array}\right\} \Longrightarrow \frac{\partial X}{\partial x} = 0 \quad \therefore X = \text{constant}$$

Lines diverge



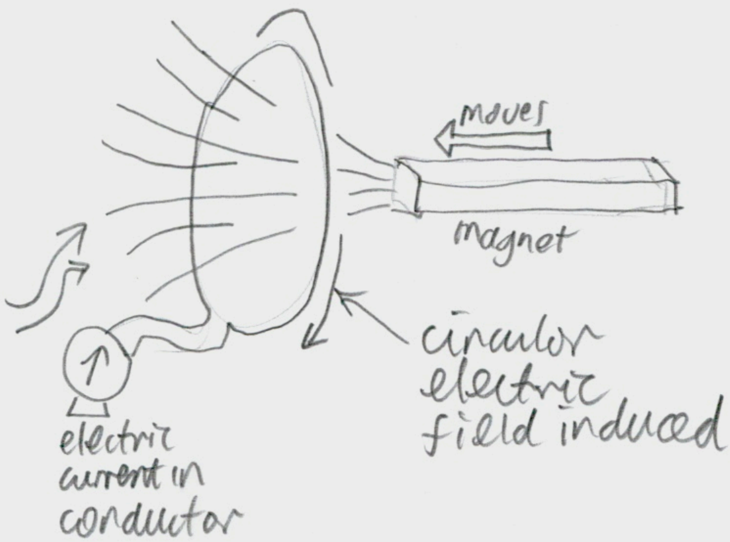
If $\frac{\partial Y}{\partial y} > 0$ $\frac{\partial Z}{\partial z} > 0 \Rightarrow \frac{\partial X}{\partial x} < 0$ $\therefore$ X weakens with increasing x

more precise grounding : Gauss' theorem "$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 0$"

$$\mathrm{II} \quad \boxed{\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} = 0}$$

magnetic lines of force conserved

Faraday induction



"Gaussian units"

$$\mathrm{III} \quad \left[\begin{aligned}
\frac{1}{c}\frac{\partial L}{\partial t} &= \frac{\partial Y}{\partial x} - \frac{\partial L}{\partial y} \\
\frac{1}{c}\frac{\partial M}{\partial t} &= \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z} \\
\frac{1}{c}\frac{\partial N}{\partial t} &= \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}
\end{aligned}\right]$$

change in time of magnetic field — circular electric field

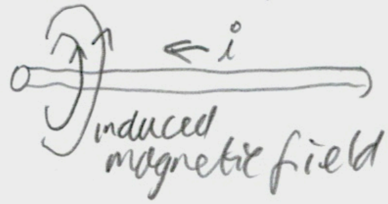
More precise grounding

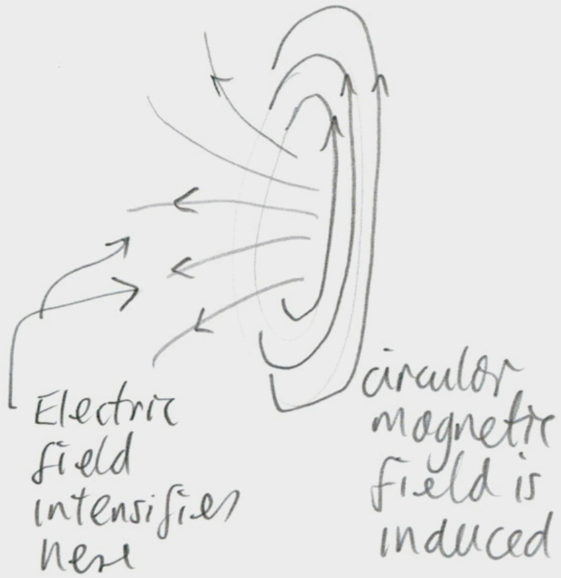
"Stokes theorem"

$$-\frac{1}{c}\frac{\partial \underset{\sim}{H}}{\partial t} = \underset{\sim}{\nabla} \times \underline{E}$$

Maxwell's "Displacement current"] — increasing electric field induces circular magnetic field

analogous to Oersted/Ampere law





IV

$$\frac{1}{c}\frac{\partial X}{\partial t} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}$$

$$\frac{1}{c}\frac{\partial Y}{\partial t} = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}$$

$$\frac{1}{c}\underbrace{\frac{\partial Z}{\partial t}}_{\text{change in time of electric field}} = \underbrace{\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}}_{\text{circular magnetic field}}$$

more precise grounding

"Stokes theorem"

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H}$$

Einstein's Project

maxwell's equations hold in $K(x,y,z,t)$ $\xrightarrow{\text{Principle of Relativity}}$ maxwell's equations hold in $k(\xi,\eta,\zeta,\tau)$

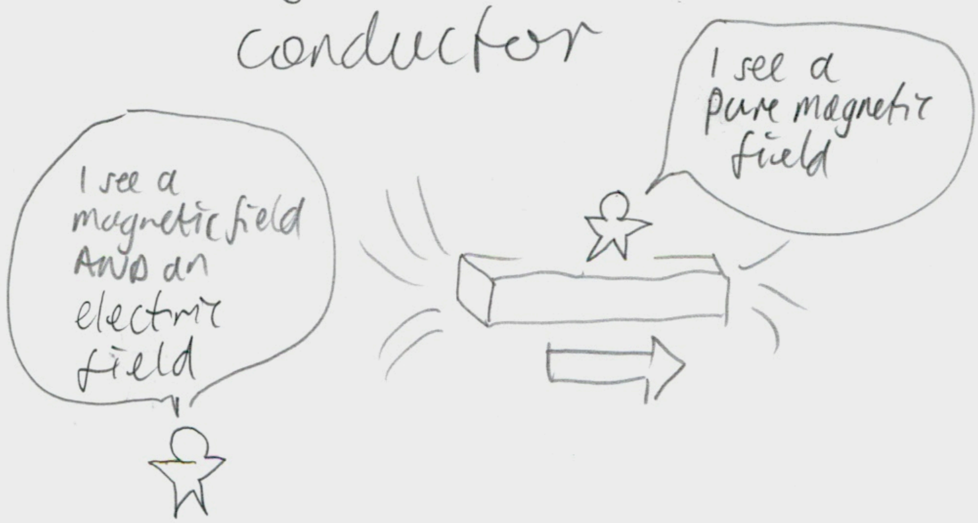
show this is so

x,y,z,t transform by Lorentz transformation

What about field quantities

X,Y,Z ?
L,M,N

magnet and conductor

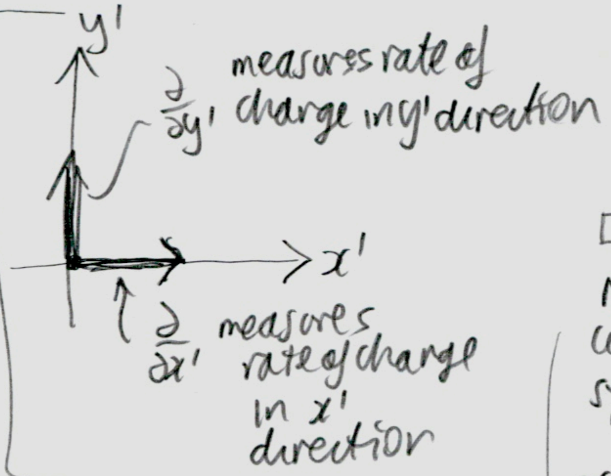


Pure magnetic field
L,M,N
$X=Y=Z=0$

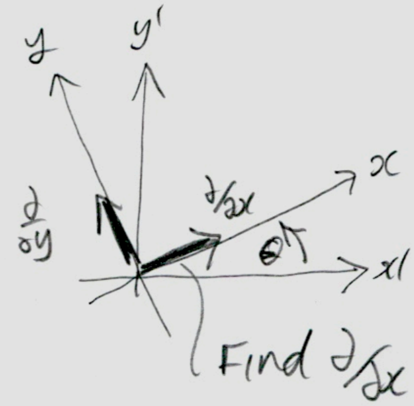
$\longrightarrow$

magnetic and electric field
L,M,N
$X,Y,Z \neq 0$

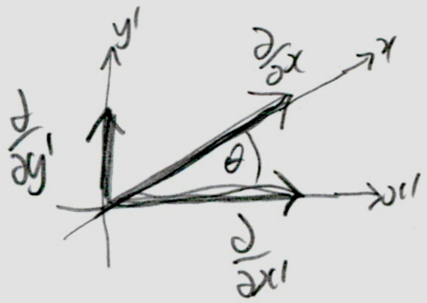
Motivating The Chain Rule for Derivative Operators



New coordinate system



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$



$\partial/\partial x$ is fixed by $\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}$

linear dependence

$$\frac{\partial}{\partial x} = \begin{pmatrix}\text{some} \\ \text{constant}\end{pmatrix}\frac{\partial}{\partial x'} + \begin{pmatrix}\text{some} \\ \text{constant}\end{pmatrix}\frac{\partial}{\partial y'}$$

$$\left.\frac{\partial x'}{\partial x}\right|_{y\,\text{constant}} = \cos\theta$$

$$\left.\frac{\partial y'}{\partial x}\right|_{y\,\text{constant}} = \sin\theta$$

$$\boxed{\frac{\partial}{\partial x} = \cos\theta\,\frac{\partial}{\partial x'} + \sin\theta\,\frac{\partial}{\partial y'}}$$

same as rule of vector decomposition

(Derivative operators $\sim$ vectors)

Preparation: Transformations of space, time derivative operators

$$\tau = \beta(t - v/c^2\, x) \qquad \xi = \beta(x - vt) \qquad \eta = y \qquad \zeta = z$$

$$\left[\frac{\partial}{\partial t} = \underbrace{\frac{\partial \tau}{\partial t}}_{\beta} \cdot \frac{\partial}{\partial \tau} + \underbrace{\frac{\partial \xi}{\partial t}}_{-\beta v} \cdot \frac{\partial}{\partial \xi} + \underbrace{\frac{\partial \eta}{\partial t}}_{0} \cdot \frac{\partial}{\partial \eta} + \underbrace{\frac{\partial \zeta}{\partial t}}_{0} \cdot \frac{\partial}{\partial \zeta} = \beta \frac{\partial}{\partial \tau} - \beta v \frac{\partial}{\partial \xi} \right]$$

$$\left[\frac{\partial}{\partial x} = \underbrace{\frac{\partial \xi}{\partial x}}_{\beta} \cdot \frac{\partial}{\partial \xi} + \underbrace{\frac{\partial \tau}{\partial x}}_{-\beta v/c^2} \cdot \frac{\partial}{\partial \tau} + \underbrace{\frac{\partial \eta}{\partial x}}_{0} \cdot \frac{\partial}{\partial \eta} + \underbrace{\frac{\partial \zeta}{\partial x}}_{0} \cdot \frac{\partial}{\partial \zeta} = \beta \frac{\partial}{\partial \xi} - \frac{\beta v}{c^2} \frac{\partial}{\partial \tau} \right]$$

$$\left[\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta} \right] \qquad \left[\frac{\partial}{\partial z} = \frac{\partial}{\partial \zeta} \right]$$

Inverses $\dfrac{\partial}{\partial \tau} = \beta \dfrac{\partial}{\partial t} + \beta v \dfrac{\partial}{\partial x}$ $\quad \dfrac{\partial}{\partial \xi} = \beta \dfrac{\partial}{\partial x} + \dfrac{\beta v}{c^2} \dfrac{\partial}{\partial \tau}$

$$\frac{1}{c}\frac{\partial \underline{E}}{\partial t} = \underline{\nabla}\times\underline{H} \qquad \underline{E}=(X,Y,Z) \quad \underline{H}=(L,M,N)$$

Maxwell's equations in $K(x,y,z,t)$ transform to $k(\xi,\eta,\zeta,\tau)$

x-component

$$\frac{1}{c}\frac{\partial X}{\partial t} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}$$

$\Downarrow$ Add $\frac{v}{c}\frac{\partial X}{\partial x}$ to both sides.

where $\frac{v}{c}\frac{\partial X}{\partial x} = -\frac{v}{c}\frac{\partial Y}{\partial y} - \frac{v}{c}\frac{\partial Z}{\partial z}$ since $\underline{\nabla}\cdot\underline{E}=0$, $\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0$

multiply by β

$$\frac{1}{c}\beta\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)X = \beta\,\frac{\partial(N-\frac{v}{c}Y)}{\partial y} - \beta\,\frac{\partial(M-\frac{v}{c}Z)}{\partial z}$$

$\Downarrow$ substitute for space, time operators

$$\frac{1}{c}\frac{\partial X}{\partial \tau} = \frac{\partial}{\partial \eta}\beta\left(N-\frac{v}{c}Y\right) - \frac{\partial}{\partial \zeta}\beta\left(M-\frac{v}{c}Z\right)$$

y-component

$$\frac{1}{c}\frac{\partial Y}{\partial t} = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}$$

$\Downarrow$ Add $-\beta^2\frac{v}{c^2}\frac{\partial N}{\partial t} + \beta^2\frac{v}{c}\frac{\partial Y}{\partial x}$ to both sides.

Insert factors $1 = \beta^2(1-v^2/c^2)$

$$\frac{1}{c}\overbrace{\beta^2\left(1-\frac{v^2}{c^2}\right)}^{1}\frac{\partial Y}{\partial t} - \beta^2\frac{v}{c^2}\frac{\partial N}{\partial t} + \beta^2\frac{v}{c}\frac{\partial Y}{\partial x} = \frac{\partial L}{\partial z} - \beta^2\frac{v}{c^2}\frac{\partial N}{\partial t} + \beta^2\frac{v}{c}\frac{\partial Y}{\partial x} - \overbrace{\beta^2\left(1-\frac{v^2}{c^2}\right)}^{1}\frac{\partial N}{\partial x}$$

$$\frac{1}{c}\beta^2\frac{\partial Y}{\partial t} - \beta^2\frac{v}{c^2}\frac{\partial N}{\partial t} + \beta^2\frac{v}{c}\frac{\partial Y}{\partial x} - \beta^2\frac{v^2}{c^2}\frac{\partial N}{\partial x} = \frac{\partial L}{\partial z} - \beta^2\frac{v}{c^2}\frac{\partial N}{\partial t} + \beta^2\frac{v^2}{c^3}\frac{\partial Y}{\partial t} - \beta^2\frac{\partial N}{\partial x} + \beta^2\frac{v}{c}\frac{\partial Y}{\partial x}$$

$$\frac{1}{c}\left(\beta\frac{\partial}{\partial t} + \beta v\frac{\partial}{\partial x}\right)\beta\left(Y-\frac{v}{c}N\right) = \frac{\partial L}{\partial z} - \beta\left(\frac{v}{c^2}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\beta\left(N-\frac{v}{c}Y\right)$$

$\Downarrow$ substitute for space, time operators

$$\frac{1}{c}\frac{\partial}{\partial \tau}\beta\left(Y-\frac{v}{c}N\right) = \frac{\partial L}{\partial \zeta} - \frac{\partial}{\partial \xi}\beta\left(N-\frac{v}{c}Y\right)$$

... etc. for remaining maxwell equations

see 5a for more details

If maxwell's equations also hold in k,
$\equiv$ then fields must transform as

in k $\rightarrow$ $\quad\quad$ $\leftarrow$ in K

$$X' = \psi(v)X \qquad L' = \psi(v)L$$
$$Y' = \psi(v)\beta(Y - \tfrac{v}{c}N) \qquad M' = \psi(v)\beta(M + \tfrac{v}{c}Z)$$
$$Z' = \psi(v)\beta(Z + \tfrac{v}{c}M) \qquad N' = \psi(v)\beta(N - \tfrac{v}{c}Y)$$

Fix $\psi(v)$ = since

- Transformation forms a group.

$$X \xrightarrow{\text{add } v} X' = \psi(v)X \xrightarrow[v]{\text{subtract}} X'' = \psi(-v)X'$$

But $X'' = X$ $\therefore$ $\psi(-v)\psi(v)X = X$

$$\boxed{\psi(v)\psi(-v) = 1}$$

- By symmetry

$$\boxed{\psi(v) = \psi(-v)}$$

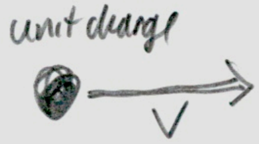
see footnote 4.
Brief new $\underline{E}$ induced from $\underline{H}$ by change frame of reference must flip direction if $v \rightarrow -v$.

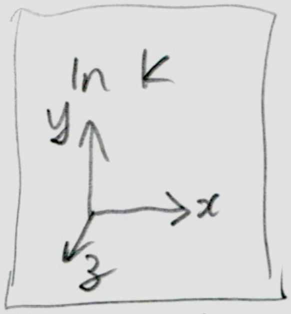
$$\psi(v)\psi(-v) = (\psi(v))^2 = 1$$

$$\psi(v) = \pm 1$$

choose plus to retain agreement in direction axes
i.e. rule out $\psi(0) = -1$

"Old manner of Expression"





Force = Electric force + magnetic force

$\underset{\sim}{E} = (X, Y, Z)$

$\frac{1}{c}\underset{\sim}{V} \times \underset{\sim}{H}$

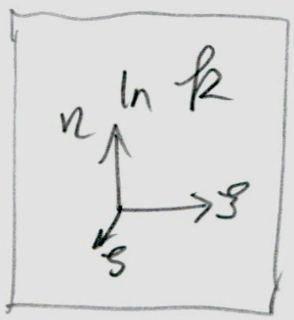
$= \left(0, -\frac{v}{c}N, \frac{v}{c}M\right)$

use determinant rule

$$\frac{1}{c}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v & 0 & 0 \\ L & M & N \end{vmatrix}$$

$\Downarrow$ simplify

"New manner of Expression"



Force = Electric force

No magnetic force since charge at rest

(X', Y', Z')

Lorentz transform

"Neglect terms... second and higher in v/c"

$\beta \approx 1$

$$X' = X$$

$$Y' = Y - \tfrac{v}{c} N$$

$$\underbrace{Z'}_{\text{Force in } k} = \underbrace{Z + \tfrac{v}{c} M}_{\text{Force in } K}$$