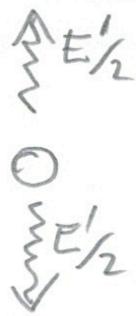
Einstein's derivation of $E=mc^2$ based on his simplified 1946 derivation

Body emits equal amounts radiant energy in opposite directs in body rest frame

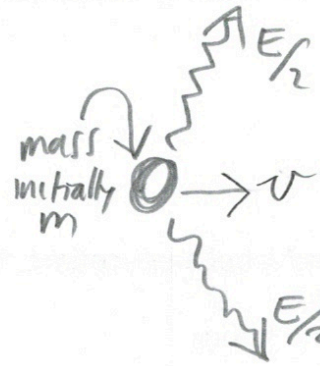


Body remains at rest by symmetry

Redescribe in frame in which body moves at v ⇒

Body emits equal amount of energy $E/2$

Propogation inclined in direction of motion



Body continues to move at v

since

Body loses energy E

light carries momentum $\frac{E/2}{c}$

component of momentum in direction of v is

$$\frac{E}{2c}\cdot\frac{v}{c}$$

Radiation carries off momentum

$$\frac{E}{2c}\cdot\frac{v}{c}+\frac{E}{2c}\frac{v}{c}$$

$$=\frac{E}{c^2}v$$

conservation mass loses this momentum

change in momentum of body $=\frac{E}{c^2}v=$ mass after emission $\times v$ − mass before emission $\times v$

change in mass $= E/c^2$

of Maxwell's electrodynamics and Lorentz's theorem of corresponding states. The only assumptions would be that starlight is a propagating waveform conforming to the principle of relativity.[18] If our concept of time was to be grounded in experience, here was experience calling for a concept of time that incorporated the relativity of simultaneity.

5. E = MC²

5.1. The Result

Shortly after completing his paper on special relativity, Einstein found another consequence of the theory that he described in a short note "Does the Inertia of a Body Depend upon its Energy Content?" (Einstein 1905s). The basic notion was as simple as it was profound. Any quantity of energy, an amount "E" for example, also carries a mass "m" in direct proportion to the energy. The mass is computed by dividing the energy E by the number c^2. That number is so large that the associated mass is usually very tiny. Conversely, any mass m is also a quantity of energy E, where the conversion is effected by multiplying m by c^2. Because c^2 is so large, even a very small mass is associated with an enormous amount of energy.

This result of the inertia of energy can be applied whenever mass or energy transforms. Sometimes the effect is an imperceptible curiosity. When we talk on a battery-powered cell phone, the battery loses energy as it powers the phone. The accompanying, miniscule loss of mass of the battery is imperceptible to us. On other occasions, the effect is world changing. When uranium-235 undergoes fission, it breaks into other elements whose total mass turns out to be slightly less than that of the original uranium. That slight mass deficit manifests as an enormous quantity of energy in heat and radiation. As was discovered decades later, that process can power atom bombs or nuclear power plants.

To the casual reader, virtually all of Einstein's demonstrations of $E = mc^2$ seem curiously complicated, drawing on arcane results in electrodynamics, now generally regarded as more obscure than the result to be shown. Even a mid-century derivation (Einstein 1946b), offered as especially simple, takes the pressure of radiation as a primitive notion. The reasons for this obliqueness lie in the physics and in its history. Special relativity, as a theory of space and time, cannot make pronouncements by itself on energy, mass, and matter. It can only constrain the ways that they can manifest in space and time: they must be governed by laws that admit no absolute velocities. So some extra physical assumption must be supplied to determine which of the possibilities is realized. In Einstein's case, that extra assumption is conveyed by electrodynamics. The choice of electrodynamics for this purpose is entirely natural. The inertia of energy is a result already to be found in Maxwell's electrodynamics, just as the kinematics of special relativity were first discovered in Maxwell's theory. The real import of Einstein's demonstrations is to show that the inertia of energy cannot be localized to electrodynamics alone. Once it is secured there, relativity theory demands that it must hold for all forms of energy.[19]

5.2. *A Demonstration*

The following is a version of Einstein's (1905s) demonstration, simplified along the lines of Einstein (1946b).[20] It is designed to show that if the inertia of energy is realized in Maxwell's electrodynamics, it must be realized for all forms of mass and energy. The inertia of energy is expressed in Maxwell's theory for unidirectional radiation as follows: a quantity of radiant energy E carries momentum E/c in the direction of its motion. (To make the result familiar, assume that momentum has magnitude mc where m is the mass of radiation and we have $E/c = mc$ so that $E = mc^2$.)

A body with mass m' at rest emits two quantities of radiant energy $E'/2$ in opposite directions, as shown in Figure 2.16.

Because of the symmetry of the emission, the body remains at rest. We now view the process from a frame in which the body moves

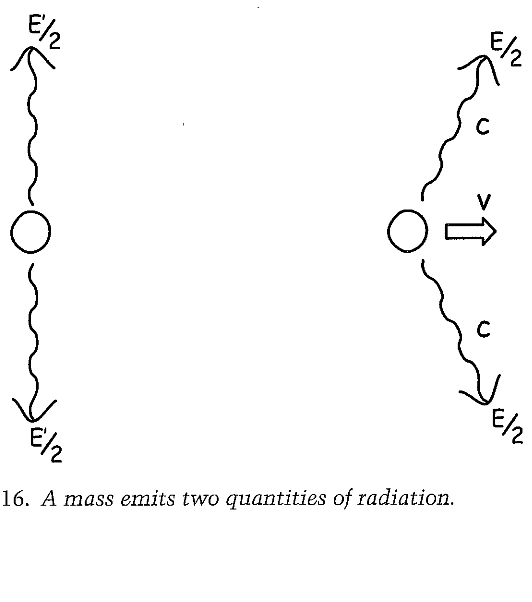


Figure 2.16. *A mass emits two quantities of radiation.*

perpendicularly to the direction of emission at v, and in which it has mass m. The quantities of radiant energy are now $E/2$ in the new frame. Thus they carry momentum $E/2c$ in the direction of propagation and a portion of that momentum in the ratio v/c lies in the direction of the body's motion. That portion is $(E/2c)(v/c) = (1/2)(E/c^2)v$. The law of conservation of momentum tells us that momentum gained by the radiation must equal that lost by the body. In the direction of the body's motion, the radiation has gained momentum $2 \times (1/2)(E/c^2)v = (E/c^2)v$. So the momentum of the body must be reduced by the same amount. The momentum of the body is mv and it must reduce by $(E/c^2)v$ as a result of the emission. Since the emission did not accelerate the body in its rest frame, the same will be true in this frame. Therefore the body's velocity remains v. So the decrease of momentum must come from a reduction in the mass m of the body. In sum, the body loses energy E and, as a result, loses momentum $(E/c^2)v$, which corresponds to a loss of mass of (E/c^2). This is the inertia of energy, now demonstrated for any body whatever that can emit radiation in the way shown.

6. CONCLUSION

The special theory of relativity owes its origins to Maxwell's equations of the electromagnetic field. (Einstein 1949a, 59)

In our brief review of the origins of Einstein's theory, we have seen much to affirm Einstein's judgment. The theory was already implicit in Maxwell's electrodynamics – so much so that Lorentz was able to discover its essential mathematical structure without realizing that he had chanced upon a new theory of space and time. On the basis of this theory, Poincaré had also begun to speak of the principle of relativity as one of the principles to which all physics must be subject. In an analysis that has excited and maddened later commentators, Poincaré interpreted Lorentz's local time in terms of the synchronizing of clocks by light signals, just as Einstein did later, but without conveying a sense that this construction was the core of a new physical theory of space and time (see Darrigol 1995, 2004).

In assessing both Lorentz's and Poincaré's work, one must guard against interpreting their thought and goals solely in terms of their proximity to Einstein's work. It is entirely possible to recognize that no experiment will reveal the Earth's motion through the ether and even to codify this expectation as a principle of relativity, without demanding that our theories be overturned so as to eradicate all trace of an ether and its state of rest. Another principle of physics, also discussed by Poincaré, illustrates this. The second law of thermodynamics assures us

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