

Cosmology 001

How to build spaces

Euclidean space

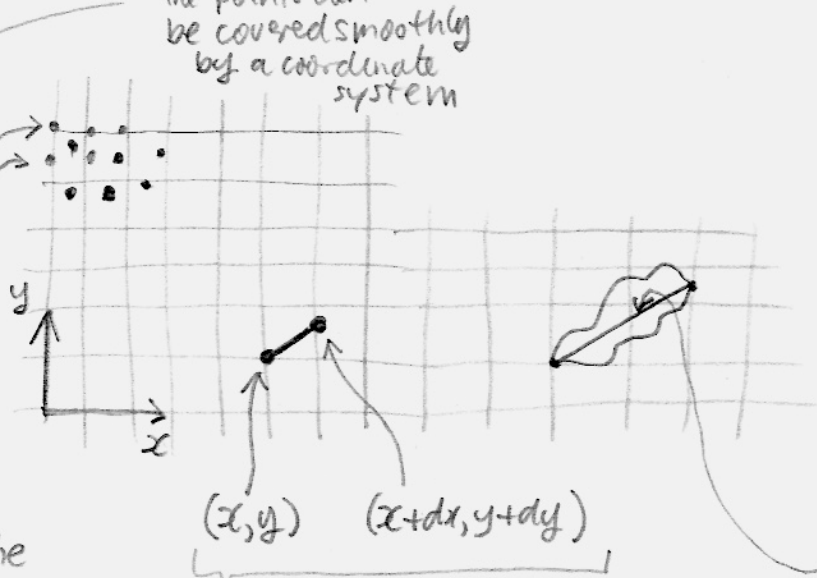
Manifold of points

This means the points can be covered smoothly by a coordinate system

+

metrical structure

Tells us the distance between points. Encoded in line element



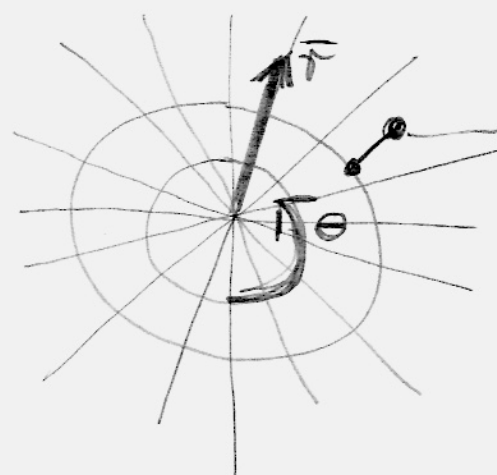
Straight lines are geodesics = curves of shortest distance.

This is the straight line!

separated by distance dl where

$$dl^2 = dx^2 + dy^2 \text{ [Pythagoras!]}$$

We can use any coordinate system we like.
e.g. radial coordinates



$$dl^2 = d\bar{r}^2 + \bar{r}^2 d\theta^2$$

Define new coordinate

$$r = \bar{r}/R$$

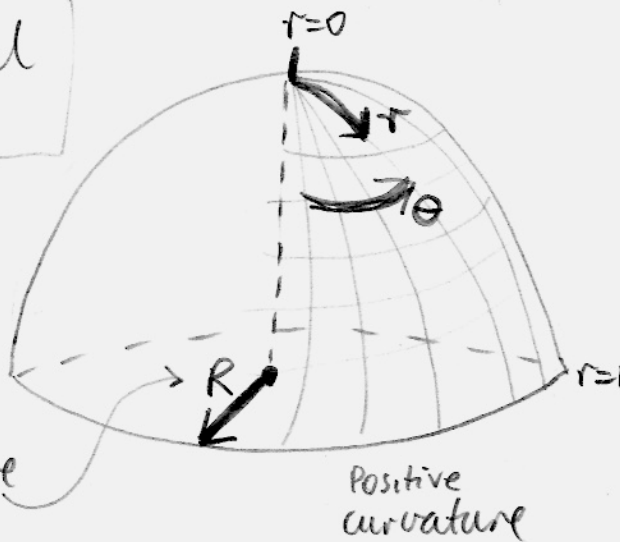
select this value arbitrarily

⇓

$$dl^2 = R^2 [dr^2 + r^2 d\theta^2]$$

Spaces of Constant curvature

Spherical space

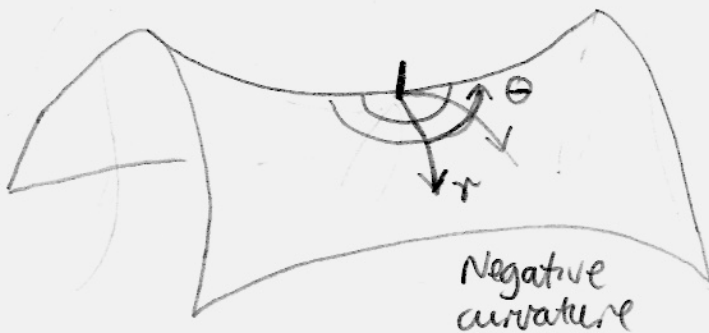


Radius of curvature

Positive curvature

$$dl^2 = R^2 \left[\frac{dr^2}{1-r^2} + r^2 d\theta^2 \right]$$

Hyperbolic space



Negative curvature

$$dl^2 = R^2 \left[\frac{dr^2}{1+r^2} + r^2 d\theta^2 \right]$$

General formula for line element

$$dl^2 = R^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 \right]$$

$k=0$ Euclidean

$k=+1$ Spherical

$k=-1$ Hyperbolic

"Straight" lines are still geodesics = curves of shortest distance

"straight" lines
are still geodesics
= curves of shortest
distance

Spaces of
variable
curvature



$$dl^2 = g_{11}(dx^1)^2 + g_{12}dx^1dx^2 + g_{21}dx^2dx^1 + g_{22}(dx^2)^2$$

$$= \sum_{i,k=1,2} g_{ik} dx^i dx^k$$

Einstein
summation
convention

$$= g_{ik} dx^i dx^k$$

variability of curvature
encoded in the way that
the g_{11}, \dots, g_{22} vary
from point to point

The matrix of
coefficients
$$g_{ik} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

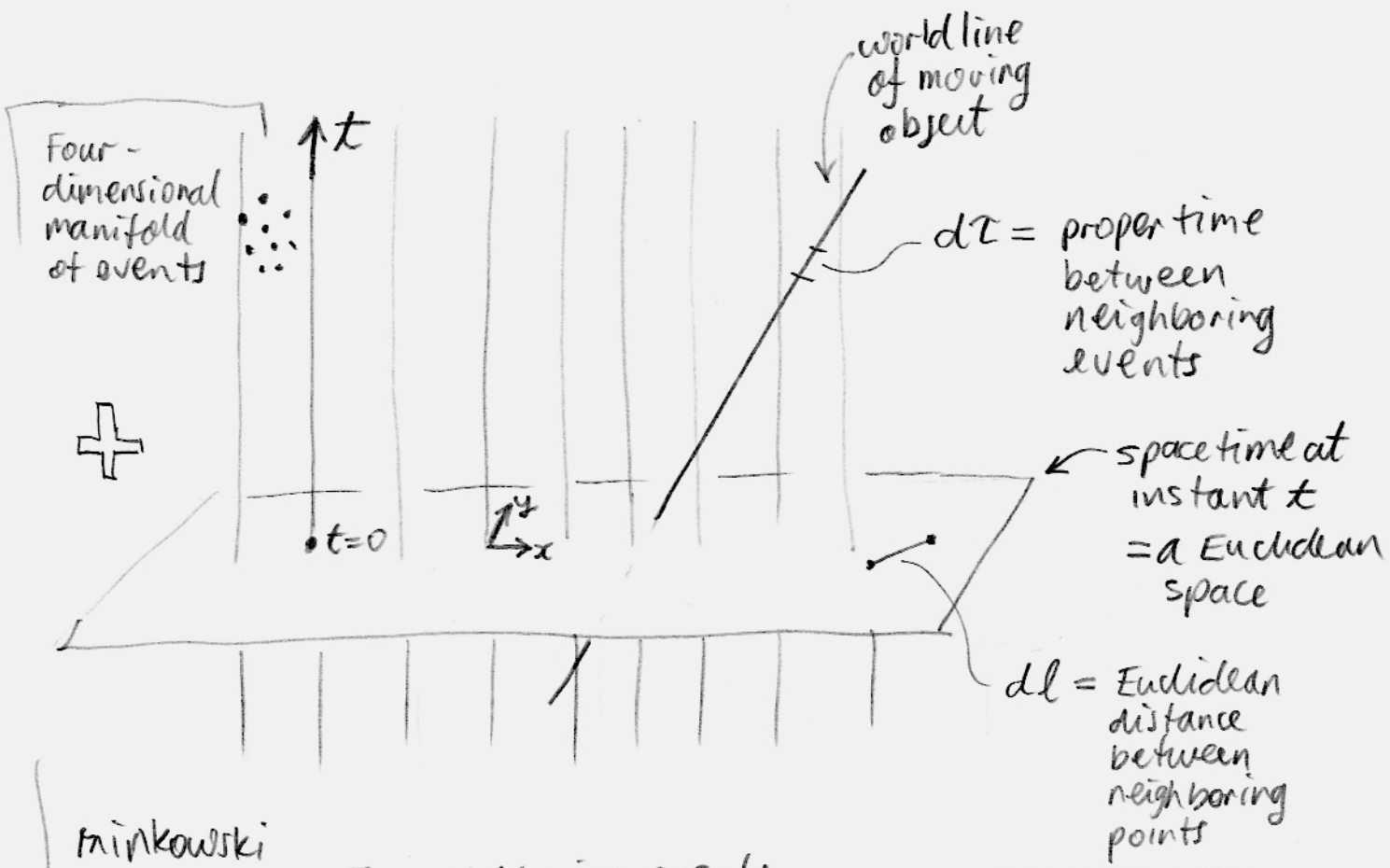
is the metric tensor
 g in the coordinate
system x^1, x^2

Example:
metric for
spherical space
$$g_{ik} = \begin{bmatrix} \frac{R^2}{1-r^2} & 0 \\ 0 & R^2 r^2 \end{bmatrix}$$

Spacetime

Minkowski spacetime = special relativity

the case of a flat spacetime



Minkowski metric

For neighboring events $(x, y, t), (x+dx, y+dy, t+dt)$
 dx, dy given via the interval ds

$$ds^2 = dt^2 - (dx^2 + dy^2) \begin{cases} ds = d\tau & \text{timelike intervals} \\ ds = -dl & \text{spacelike intervals} \end{cases}$$

$c=1$

Minkowski's big discovery!

Inertial trajectories = Free fall trajectories = Geodesics = curves of extremal interval

In this case greatest

General Relativity: The core idea

old view: Gravitational force deflects bodies into sun (Newton)

↑ time
Far from the sun, spacetime is close to Minkowskian

Free fall trajectories bend towards sun

Free fall trajectories



New view: spacetime near sun is (metrically) curved. Bodies in free fall still follow geodesics, but these geodesics curve towards sun. Farewell to gravitational force
Einstein

Einstein Gravitational Field Equations

Fix metric in $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ spacetime coordinates x^1, \dots, x^4
 Represents time, spatial geometry and gravitation



Einstein tensor represents curvature of spacetime $G_{\mu\nu}$

$$= \frac{8\pi K}{\text{constant}} T_{\mu\nu}$$

"stress-energy tensor" represents energy, momentum & stresses $T_{\mu\nu}$

(1915)

But not enough wiggle room. So Einstein (1917) added a term

Important since stresses generate gravitational effects!

$$G_{\mu\nu} + \lambda g_{\mu\nu} = \frac{8\pi K}{\text{constant}} T_{\mu\nu}$$

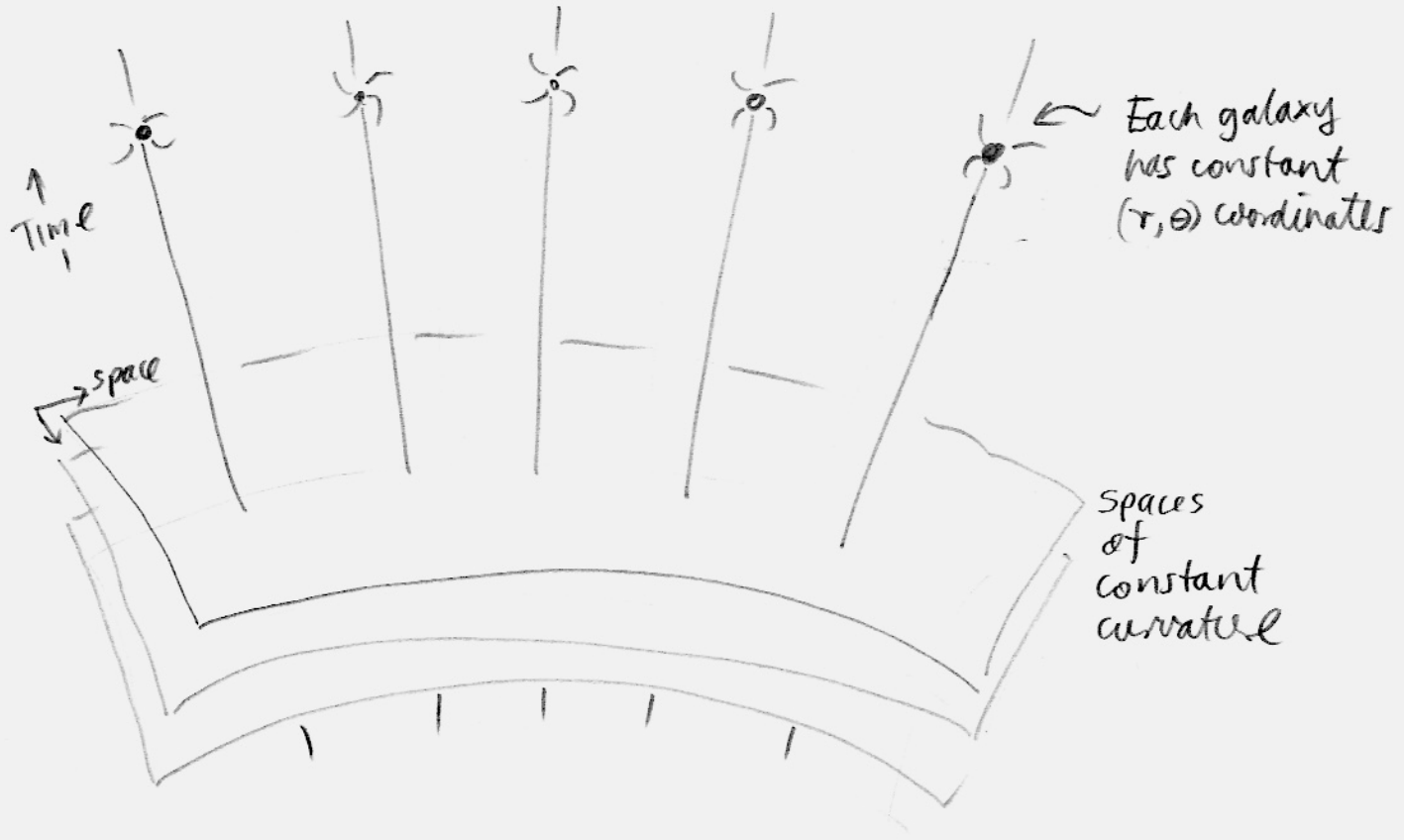
undetermined "cosmological constant"

sign conventions from Misner, Thorne & Wheeler.	Henceforth set $K=1$ $C=1$
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Robertson-Walker spacetimes = simplest relativistic cosmologies

General cosmology with homogeneous, isotropic spaces

used in 97.4% of the cosmology literature



Line element $ds^2 = dt^2 - R^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 \right]$

Functional dependence of R on t = complete dynamical history of the universe

e.g. R grows with t

galaxies get further apart

universe expands

Finding this is 73.6% of the work in cosmology

Recover dynamics for R :

Einstein field equations

$$G_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

□

specialize to case of

* Robertson-Walker spacetime

* Spatially homog., isotropic matter distribution

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$$

Rest energy density of smoothed out matter

pressure of matter

Four velocity of matter

much easier than you thought !!!

Matter couples into cosmic dynamics merely by fixing values for ρ, p .

simple cases:

(I) Very slow moving dust (galaxies, particles of non-zero rest mass: protons, neutrons)

$\rho =$ rest energy density

$p = 0$

(II) Radiation

(cosmic microwaves, em fields, neutrinos etc.)

energy density

$$\rho = 3p$$

radiation pressure

Equivalent to assertion

"Radiation has zero rest mass"

Basic equations of standard cosmological model:

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

$$\left(\frac{\dot{R}}{R}\right)^2 = -\frac{k}{R^2} + \frac{8\pi}{3}\rho + \frac{\Lambda}{3}$$

" $\dot{\bullet}$ " = $\frac{d}{dt}$

Read off dynamics: Case of $\Lambda = 0$

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}(\rho + 3p)$$

$$\left(\frac{\dot{R}}{R}\right)^2 = -\frac{k}{R^2} + \frac{8\pi\rho}{3}$$

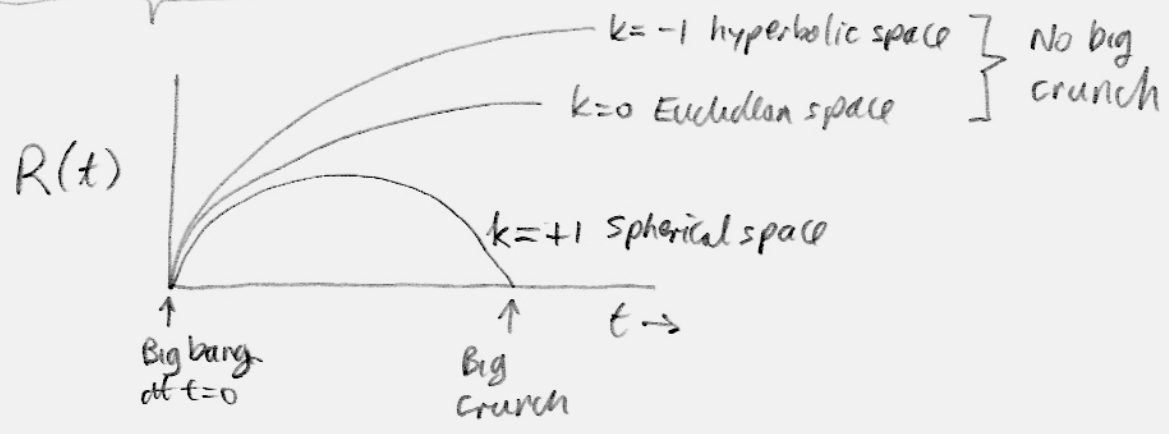
$\rho + 3p > 0$
 $\therefore \ddot{R}$ always < 0

\therefore Galaxies always accelerating towards each other

No static cosmology

$k > 0$: \dot{R} can drop to zero. Cosmic expansion can halt & recollapse begins

$k \leq 0$: \dot{R} cannot change sign. Cosmic expansion cannot halt



Case of $\Lambda \neq 0$

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

Exerts repulsive "force" "negative pressure"
 Causes galaxies to accelerate away from each other

Static cosmologies possible

$$\ddot{R} = 0 \Rightarrow \Lambda = 4\pi(\rho + 3p)$$

Table 5.1
DUST AND RADIATION FILLED ROBERTSON-WALKER COSMOLOGIES

Case of $\Lambda = 0$	TYPE OF MATTER	
	"Dust" $P = 0$	Radiation $P = \frac{1}{3}\rho$
Spatial Geometry		
3-sphere, $k = +1$	$a = \frac{1}{2}C(1 - \cos \eta)$ $\tau = \frac{1}{2}C(\eta - \sin \eta)$	$a = \sqrt{C'} [1 - (1 - \tau/\sqrt{C'})^2]^{1/2}$
Flat, $k = 0$	$a = (9C/4)^{1/3} \tau^{2/3}$	$a = (4C')^{1/4} \tau^{1/2}$
Hyperboloid, $k = -1$	$a = \frac{1}{2}C(\cosh \eta - 1)$ $\tau = \frac{1}{2}C(\sinh \eta - \eta)$	$a = \sqrt{C'} [(1 + \tau/\sqrt{C'})^2 - 1]^{1/2}$

Wald's $a \leftrightarrow R$ here
 $\tau \leftrightarrow t$

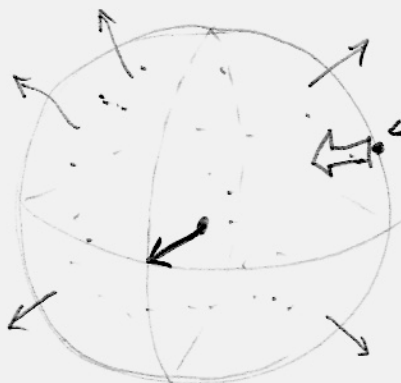
η is a parameter
 that couples
 a, τ

From
 R.M. Wald
 General
 Relativity

Visualize Dynamics via Newtonian Analogy

Milne, McCrea 1934

Sphere of dust expands



Sphere of radius $R(t)$

Expansion decelerated by inward pull of gravity according to Newton's inverse square law measured by \ddot{R}

Recover exactly the equations of the relativistic model for R !!

Newton's inverse square law

$$\implies \ddot{R} = -\frac{4\pi}{3} \rho R$$

Acceleration of unit mass on edge of sphere
 $= -(\text{mass of sphere}) / R^2$
 $= -\frac{4\pi}{3} \rho R^3 / R^2$

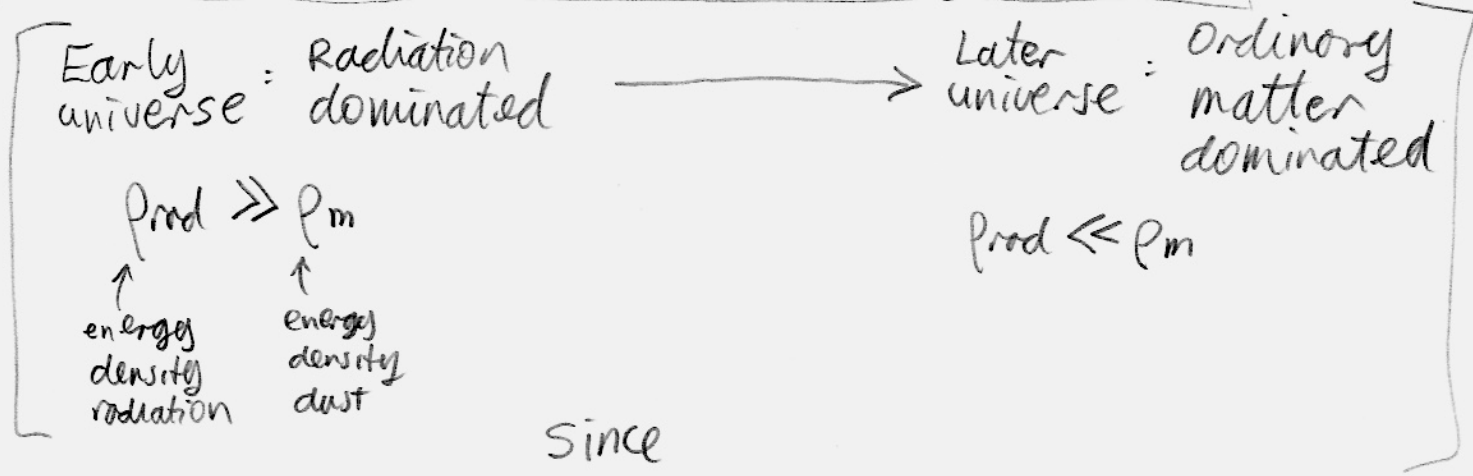
Conservation of energy

$$\implies \underbrace{\frac{1}{2} \dot{R}^2}_{\text{kinetic energy}} - \underbrace{\frac{4\pi}{3} \rho R^2}_{\text{potential energy}} = \underbrace{-\frac{k}{2}}_{\text{constant}}$$

$k=0, -1 \rightarrow$ Total energy non-negative \rightarrow cloud has sufficient kinetic energy to climb out of potential well \rightarrow no re-collapse

$k=+1 \rightarrow$ Total energy negative \rightarrow cloud does not have sufficient energy to climb out of potential well \rightarrow re-collapse

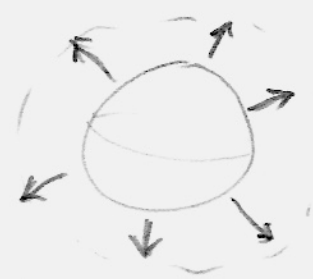
Balance of radiation and ordinary matter



During expansion ρ_m dilutes as $1/R^3(t)$
 ρ_{rad} dilutes as $1/R^4(t)$

Why?
 \Downarrow

Pick some volume $V(t)$ that moves with the cosmic expansion



In all geometries $V(t) \propto R^3(t)$

ρ_m reduces solely because its energy is distributed over a greater volume
 i.e. $\frac{d}{dt}(\rho_m V) = 0 \Rightarrow \rho_m \sim \frac{1}{V} \sim \frac{1}{R^3}$
Energy conservation

ρ_{rad} reduces because its energy is distributed over a greater volume AND its radiation pressure does lots of work during the expansion

i.e. Energy conservation: $0 = \underbrace{\frac{d}{dt}(\rho_{\text{rad}} V)}_{\text{dilution}} + \underbrace{\rho_{\text{rad}} \frac{dV}{dt}}_{\text{pressure work}} = V \frac{d\rho_{\text{rad}}}{dt} + \frac{4}{3} \rho_{\text{rad}} \frac{dV}{dt}$

$p = \frac{1}{3} \rho_{\text{rad}}$

solve: $\rho_{\text{rad}} \propto V^{-4/3} \propto 1/R^4$

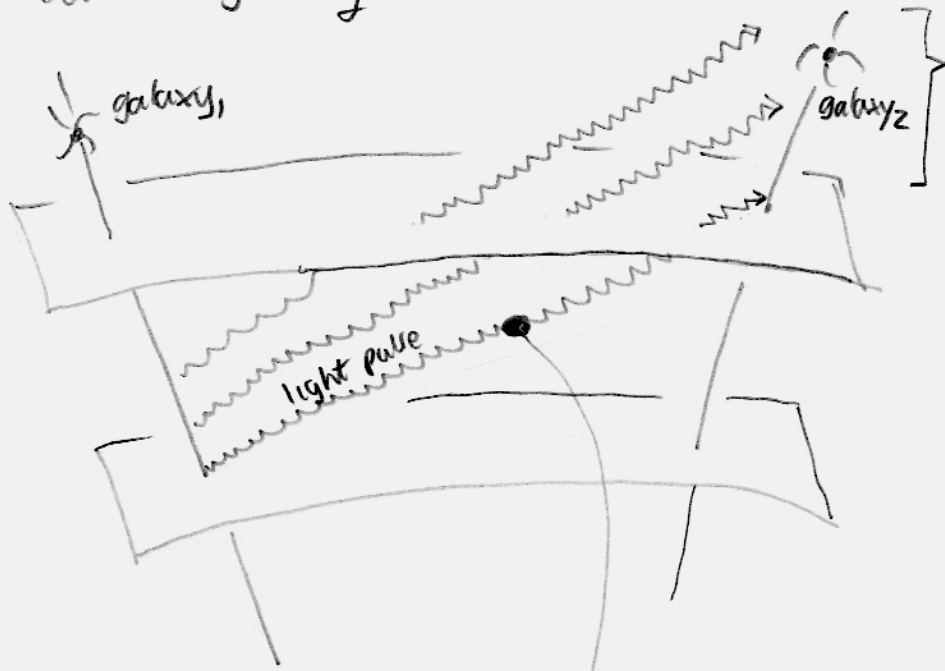
Observational foundations of modern cosmology

(two of them!)

(I) Hubble red shift

Red shift in light from distant galaxy \propto

Distance to galaxy



Expansion of space
↓
successive pulses of light from galaxy₁ must travel further to reach galaxy₂
↓
greater delay in arrival later pulses
↓
reduced frequency = red shift

(II) Cosmic microwave radiation at 2.7K (Penzias+Wilson)

Diluted remnant of high radiation density in early universe

Photon
 $Energy = h \cdot frequency$

this reduces because of Hubble red shift (due in turn to cosmic expansion)

\therefore this reduces too!

Reduction corresponds to "pressure work" loss

Add in volume dilution of photons too!

Appendix to Cosmology 001: Critical Density

One equation for dynamics of RW spacetime

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{-k}{R^2} + \frac{8\pi\rho}{3} = H^2$$

Hubble's constant² since distance L between neighboring galaxies grows as $\frac{dL}{dt} = \frac{d}{dt} \left(\frac{L}{R} \cdot R \right) = \frac{\dot{R}}{R} L = HL$
constant since expansion follows $R(t)$



$$\frac{k}{R^2} = \frac{8\pi\rho}{3} - H^2 = H^2 \left(\frac{8\pi\rho}{3H^2} - 1 \right) = H^2 \left(\frac{\rho}{\rho_{crit}} - 1 \right) \quad \text{where} \quad \rho_{crit} = \frac{3H^2}{8\pi}$$

Read off directly:

$$\rho = \rho_{crit}$$



$k=0$
Euclidean

$$\rho > \rho_{crit}$$



$k=1$
Spherical

$$\rho < \rho_{crit}$$

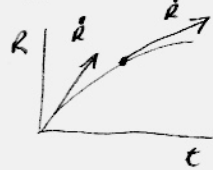


$k=-1$
Hyperbolic

(expansion)
Flatness problem: During normal time evolution, if $\rho_{crit} \neq 1$, ρ/ρ_{crit} moves to values ever further away from 1

From above $\frac{\rho}{\rho_{crit}} = \frac{k}{H^2 R^2} + 1 = 1 + \frac{k}{(\dot{R})^2}$

$(\dot{R}(t))^2$ decreases with t in expansion



$\therefore \frac{1}{(\dot{R}(t))^2}$ increases with t during expansion

If at some time t

($k=0$)

$$\frac{\rho}{\rho_{crit}} = 1$$



$\frac{\rho}{\rho_{crit}}$ remains = 1

($k=1$)

$$\frac{\rho}{\rho_{crit}} > 1$$



$\frac{\rho}{\rho_{crit}}$ grows larger

($k=-1$)

$$\frac{\rho}{\rho_{crit}} < 1$$



$\frac{\rho}{\rho_{crit}}$ grows smaller

Exotic matter Needed to Solve Flatness Problem

(as is done in Inflationary cosmology)
 $[\lambda=0]$

Normal matter:
 $\rho + 3p > 0$

Exotic matter
 $\rho + 3p < 0$

Einstein field equations

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}(\rho + 3p)$$

$\ddot{R} < 0$

$\ddot{R} > 0$

$\dot{R} > 0$ but decreasing
 in expansion phase

$\dot{R} > 0$ and increasing
 in expansion phase

$\Omega = \frac{\rho}{\rho_{critical}} = 1 + \frac{k}{\dot{R}^2}$

$\Omega = \frac{\rho}{\rho_{critical}} = 1 + \frac{k}{\dot{R}^2}$

DIVERGES
 from $\Omega = 1$
 in expansion

CONVERGES
 to $\Omega = 1$
 in expansion

