Newtonian Cosmology

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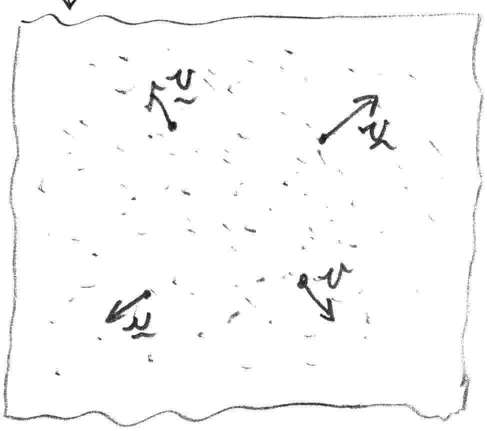
HPS 2580 Cosmology
January 2018

What is supposed

Infinite Euclidean space

Filled with uniform matter distribution of density $\rho(t)$



matter moves in a velocity field

$\mathbf{v}(\mathbf{r})$

velocity at position

Apply homogeneity isotropy cosmological principle (observers at all places see the same velocity field)

observer at $\mathbf{r}=0$

uniform expansion or contraction around $\mathbf{r}=0$

$$\boxed{\mathbf{v}(\mathbf{r},t) = \underbrace{H(t)}_{\text{"Hubble constant"}}\,\mathbf{r}}$$

"Hubble constant"

constant over $\mathbf{r}$, but varies with time t.

check compatibility with cosmological principle

Relocate observer to new position $\mathbf{r}=\mathbf{a}$

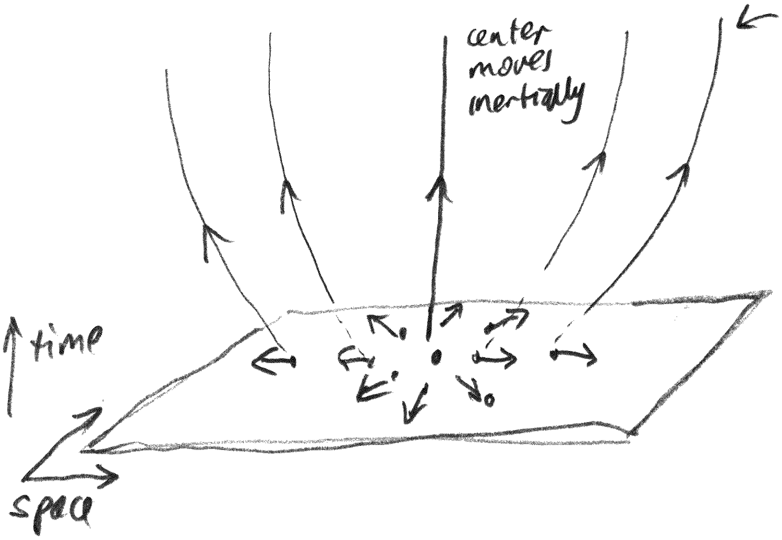
velocity field relative to $\mathbf{r}=\mathbf{a}$ is $\mathbf{v}(\mathbf{r}) - \mathbf{v}(\mathbf{a}) = H(t)\mathbf{r} - H(t)\mathbf{a}$

take off observer's velocity

$$= \underbrace{H(t)(\mathbf{r}-\mathbf{a})}_{\text{Relocated observer finds same velocity field.}}$$

Relocated observer finds same velocity field.

A center of the Universe ?



Observables of cosmology are
- Relative velocities, accelerations between mass points;
- matter density

⟹ Preserved under a shift of center to a new location

Gauge freedom of Newtonian cosmolog

"Relativity of acceleration"

For more, see papers by David Malament, John D. Norton

e.g. Norton, "The Force of Newtonian Cosmology: Acceleration is relative" Philosophy of Science 62 (1995), pp. 511-22

Recover cosmic dynamics by tracking motion of a unit test mass

"$\bullet$" $= d/dt$

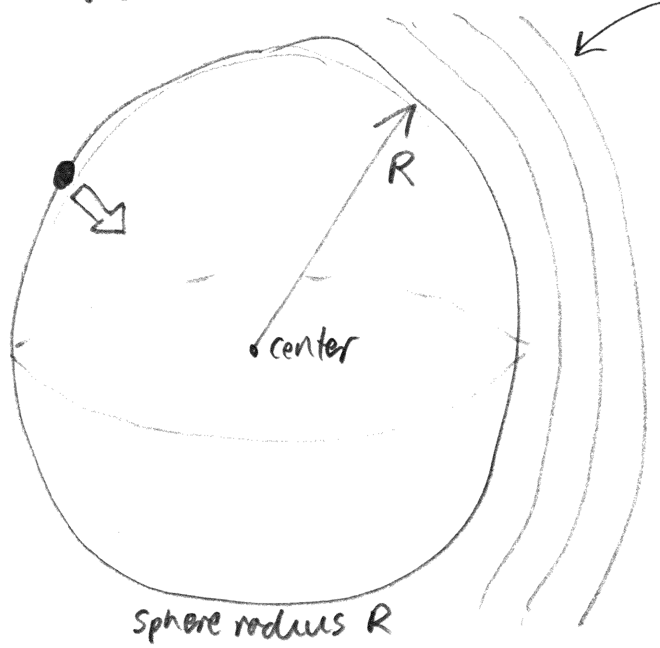
Force on unit test mass:

$$\ddot{R}(t) = -\frac{GM}{R(t)^2}$$

M = mass in sphere

$$= \frac{4\pi R^3(t)\rho(t)}{3}$$

Potential energy $= -\frac{GM}{R(t)}$



Three governing equations

Acceleration

$$\boxed{\text{I. } \ddot{R}(t) = -\frac{GM}{R(t)^2}}$$

Energy

Total energy U = kinetic energy + Potential energy

$\frac{1}{2}\dot{R}^2$ $\qquad -\frac{GM}{R}$

$$\boxed{\text{II. } \dot{R}^2 = 2U + \frac{2GM}{R}}$$

U is constant in time

conservation of mass

$$\boxed{\text{III } \dot{M} = 0}$$

mass enclosed in sphere unchanged with time as sphere grows or shrinks

Any two of these entail the third.

$$\boxed{\text{For example: } \mathrm{II} \,\&\, \mathrm{III} \Rightarrow \mathrm{I}}$$

II. $\dot{R}^2 = 2u + \dfrac{2GM}{R}$ III $\dot{M} = 0$

$d/dt = \text{"}\cdot\text{"}$

$$2\dot{R}\ddot{R} = 2GM \cdot \frac{-\dot{R}}{R^2} + \cancelto{0}{\frac{2G\dot{M}}{R}}$$

divide by $\dot{R} \neq 0$

I. $\ddot{R} = -\dfrac{GM}{R^2}$

Orbital analogs of cosmic dynamics

$u<0$

test mass does not have enough energy to escape attraction of central mass M

Elliptical orbit

collapse ellipse to maximum eccentricity

Big bang with big crunch

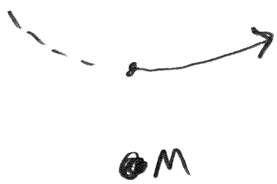
From II, $\dot{R}=0$ when

$$0 = 2u + \frac{2GM}{R}$$

There is an R that solves this since $u<0$

Intermediate case

$u=0$



parabolic orbit

big bang, no big crunch

$u>0$

Test mass has enough energy to escape attraction of central mass M

hyperbolic orbit

collapse to case of extremal eccentricity

Big bang with no big crunch

From II

$$\dot{R}^2 = 2u + \frac{2GM}{R}$$

$\dot{R}(0)>0 \Rightarrow \dot{R}(t)$ can never change sign when $u>0$

$u=0$ "Parabolic" case has a simple solution
(others have parametric solutions)

From II $\dot{R}^2 = \dfrac{2GM}{R} \Longrightarrow R(t) = \left(\tfrac{3}{2}\sqrt{2GM}\right)^{\frac{2}{3}} t^{\frac{2}{3}}$

$$\dot{R} = \frac{\sqrt{2GM}}{R^{\frac{1}{2}}}$$

$$\therefore \underbrace{R^{\frac{1}{2}}\dot{R}}_{\frac{2}{3}\underbrace{\frac{dR^{\frac{3}{2}}}{dR}\cdot\frac{dR}{dt}}_{\frac{d}{dt}\left(R^{\frac{3}{2}}\right)}} = \sqrt{2GM} \qquad \therefore \frac{d}{dt}\left(R^{\frac{3}{2}}\right) = \frac{3}{2}\sqrt{2GM}$$

$$R(t)^{\frac{3}{2}} = \frac{3}{2}\sqrt{2GM}\, t \text{ , using } R(0)=0$$

Critical density & the Hubble constant

Defined: that density of matter ρ_c now that corresponds to $U=0$

From II. for $U=0$

$$\dot{R}^2 = \frac{2GM}{R} = \frac{8\pi G}{3}\rho_c \frac{R^3}{R}$$

$$\therefore \boxed{\frac{8\pi G}{3}\rho_c = \left(\frac{\dot{R}}{R}\right)^2 = H^2}$$

Hubble constant

since Hubble law is $V = \dot{R} = HR$

i.e. critical density is independent of the size of the matter sphere considered.

Add cosmologized constant Λ

Equations become

$$\text{I.} \quad \ddot{R}(t) = -\frac{GM}{R(t)^2} + \frac{\Lambda}{3} R(t)$$

Λ adds a force of repulsion that accelerates $R(t)$

$$\text{II} \quad \dot{R}(t)^2 = 2U + \frac{2GM}{R} + \frac{\Lambda}{3} R(t)^2$$

$$\text{III} \quad \dot{M} = 0$$

Any two entail the third

(e.g. II & III $\Rightarrow$ I by slight variant of calculation on p.4)

Einstein's conception: Tune Λ to produce a static universe $\ddot{R} = 0$

$$0 = -\frac{GM}{R_{stat}^2} + \frac{\Lambda}{3} R_{stat}$$

solve

$$\Lambda = \frac{3GM}{R_{stat}^3} = \frac{3G}{R_{stat}^3} \cdot \frac{4\pi \rho R_{stat}^3}{3} = 4\pi G \rho$$

Instability of static universe:

$R = R_{stat} \Rightarrow \ddot{R}(t) = 0$

$R < R_{stat} \Rightarrow \ddot{R}(t) < 0$ and dynamics leads to $R \ll R_{stat}$

$R > R_{stat} \Rightarrow \ddot{R}(t) > 0$ " " " " $R \gg R_{stat}$

since $\left.\frac{\partial}{\partial R}\right|_t \ddot{R} = \frac{3GM}{R^3} + \Lambda > 0$ for all R, so $\ddot{R}$ changes sign from negative to positive as we pass from $R < R_{stat}$ to $R > R_{stat}$

De Sitter-like universe. When R grows very large

$$GM/R^2 \ll \frac{\Lambda}{3} R(t)$$

$$\ddot{R}(t) \approx \frac{\Lambda}{3} R(t)$$

$$R(t) \approx R(0) \exp\left(\sqrt{\frac{\Lambda}{3}}\, t\right)$$

Early universe inflation

Late universe dominated by $\Lambda \approx$ dark energy

Transition to relativistic equations of FLRW cosmology.

Replace M by $\frac{4\pi}{3}R^2\rho$. $R(t)$ is cosmological scale factor.

$$\text{I.}\quad \ddot{R} = -\frac{GM}{R^2} + \frac{\Lambda}{3}R$$

$$= -\frac{4\pi G\rho}{3} \cdot \frac{R^3}{R^2} + \frac{\Lambda}{3}R \rightarrow \boxed{\text{I.}\quad \frac{\ddot{R}}{R} = -\frac{4\pi G\rho}{3} + \frac{\Lambda}{3}}$$

$$\text{II.}\quad \dot{R}^2 = 2u + \frac{2GM}{R} + \frac{\Lambda}{3}R^2 \rightarrow \boxed{\text{II.}\quad \left(\frac{\dot{R}}{R}\right)^2 = H^2 = \frac{8\pi G\rho}{3} + \frac{2u}{R^2} + \frac{\Lambda}{3}}$$

$$= \frac{8\pi G}{3}\rho\frac{R^3}{R} + 2u + \frac{\Lambda}{3}R^2$$

$-2u \longleftrightarrow$ geometric k

$$\text{III.}\quad 0 = \dot{M} = \frac{d}{dt}\left(\frac{4\pi}{3}\rho R^3\right) \rightarrow \boxed{\text{III.}\quad \dot{\rho} + 3\rho\frac{\dot{R}}{R} = 0}$$

$$= \frac{4\pi}{3}\left(\dot{\rho}R^3 + 3\rho R^2\dot{R}\right)$$

These are almost the dynamical equations of FLRW cosmology.

In relativity, stresses can have gravitational effects.

eg. normal isotropic pressure p

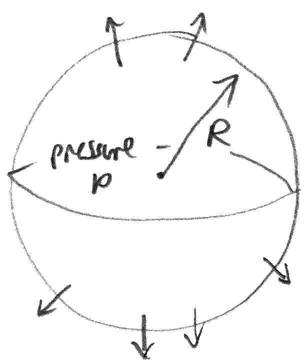
Add pressure terms to I & III with hybrid Newtonian/relativistic adjustment

Hybrid relativistic/ Newtonian theory

Post: $\underbrace{\text{Energy change}}_{\text{Pressure of expanding volume does work}} = \text{Mass change} \cdot c^2$

volume loses mass

↳ volume loses energy



Expanding sphere

work done in expanding by dV $= p\,dV =$ Energy lost by sphere $=$ mass lost $\cdot\, c^2$

$$\therefore \underbrace{\frac{d}{dt}}_{\text{compute directly}} mc^2 = -p\frac{dV}{dt} = -p\,\frac{4\pi}{3}\frac{d}{dt}R^3 = -4\pi p R^2 \dot{R}$$

$$c^2 \frac{d}{dt}\left(\frac{4\pi}{3}R^3 \rho\right) = c^2 \frac{4\pi}{3}\left(\dot{\rho}R^3 + 3\rho R^2 \dot{R}\right)$$

combine

$$\boxed{\text{III.}\quad \dot{\rho} + \frac{3\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0}$$

New term

combine with II to derive new I

stress correction

$$\boxed{\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}}$$

See over for details

$$\boxed{\begin{array}{l} \mathrm{II} \quad \dot{R}^2 = \frac{8\pi G}{3}\rho R^2 + 2u + \frac{\Lambda}{3}R^2 \\ \qquad \& \qquad \mathrm{III} \quad \dot{\rho} = -\frac{3\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) \end{array}}$$

Take d/dt of II   Assume $\frac{d}{dt}u = 0$ (why?... it works)

$$2\dot{R}\ddot{R} = \frac{8\pi G}{3}\rho 2R\dot{R} + \frac{8\pi G}{3}R^2\dot{\rho} + \frac{2\Lambda}{3}R\dot{R}$$

substitute with III

$$= \frac{8\pi G}{3}\rho 2R\dot{R} + \underbrace{\frac{8\pi G}{3}R^2\left(-\frac{3\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right)\right)}_{-\frac{8\pi G}{3}\cdot 3R\dot{R}\left(\rho + \frac{p}{c^2}\right)} + \frac{2\Lambda}{3}R\dot{R}$$

$$= \frac{8\pi G}{3}R\dot{R}\left(2\rho - 3\rho - 3\frac{p}{c^2}\right) + \frac{2\Lambda}{3}R\dot{R}$$

divide by $2\dot{R} \neq 0$

$$\ddot{R} = \frac{4\pi G}{3}R\left(-\rho - 3\frac{p}{c^2}\right) + \frac{\Lambda R}{3}$$

$$\boxed{\mathrm{I.} \quad \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}}$$

Collected hybrid equations are

I. $$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}$$

II. $$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho + 2u + \frac{\Lambda}{3}$$

III. $$\dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0$$

Hence Λ behaves like a negative pressure $p_\Lambda < 0$

$$\frac{\Lambda}{3} = -\frac{4\pi G}{3}\left(\frac{3p_\Lambda}{c^2}\right)$$

This is a purely relativistic effect