Trying to make sense of the

C P T

Theorem

Basic Structure of Quantum Field Theory (for physicists)

CPT Theorem

I Characterize the fields of interest by writing down a LORENTZ-INVARIANT Lagrangian

e.g. Klein-Gordon massive scalar field

$$\mathcal{L}[\phi(x)] = \frac{\partial \phi^\dagger}{\partial x_\mu} \cdot \frac{\partial \phi}{\partial x_\mu} - m^2 \phi^\dagger \phi$$

Looks like classical & ordinary qm Lagrangian. But these are field operators

ϕ acting on universal quantum state creates scalar particles

$\phi^\dagger$ acting on universal quantum state annihilates scalar particles

Add interaction terms

$$\underbrace{\phi^{(1)\dagger}(x)}_{\text{annihilates type (1) particle}} \underbrace{\phi^{(2)}(x)}_{\text{creates type (2) particle}}$$

II Insert Lagrangian into LORENTZ COVARIANT theoretical apparatus that spells out which processes can happen with the fields

Hamiltonian from Lorentz covariant Euler-Lagrange Equation

stories of particle creation & annihilation from perturbation theory

Vacuum ...

Lagrangians built from short list of Lorentz covariant quantities

Scalar field	Vector field	Tensor field	Pseudo tensor field	Pseudo scalar field	
$\phi(x)$	$\phi_\mu(x)$	$\phi_{\mu\nu}$	$\chi_\mu(x)$	$\chi(x)$	$\psi(x)$ Four component spinor field

$\chi_\mu(x)$, $\chi(x)$: "Pseudo" since they change sign under parity transformation

$\psi(x)$: Most fundamental since all rest can be built from these

Discrete Symmetries of Lorentz Group

Parity P : Switch left-right $(x,y,z) \longrightarrow (-x,-y,-z)$

Time reversal T : Switch future-past $t \longrightarrow -t$

Charge conjugation

C : Switch +ve and −ve charges
(and similarly for all charge-like quantities in quantum fields)

Action of C, P, T on each term that may appear in a Lagrangian...

...is determined by very long, tedious & essentially opaque computation

From R.G. Sachs, The Physics of Time Reversal
University of Chicago, 1987
pp. 157, 165, 166

As the first step, then, we write down the well-known bilinear covariants of the spinor fields for kinematically independent spinor fields $\psi^{(j)}(x)$, with $j = 1, 2, 3 \ldots$. There are just five of them, and they correspond to the tensors (I) to (V) as classified in section 6.1:

Quantities defined in terms of spinor ψ

(8.9a) $$S^{(j,k)} = :\bar{\psi}^{(j)}(x)\psi^{(k)}(x):, \quad \text{scalar}$$

(8.9b) $$V_\mu^{(j,k)} = i:\bar{\psi}^{(j)}(x)\gamma_\mu\psi^{(k)}(x):, \quad \text{four-vector}$$

(8.9c) $$T_{\mu\nu}^{(j,k)} = :\bar{\psi}^{(j)}(x)\sigma_{\mu\nu}\psi^{(k)}(x):,$$
antisymmetric tensor of second rank

(8.9d) $$(PV)_\mu^{(j,k)} = i:\bar{\psi}^{(j)}(x)\gamma_\mu\gamma_5\psi^{(k)}(x):, \quad \text{pseudovector}$$

(8.9e) $$P^{(j,k)} = i:\bar{\psi}^{(j)}(x)\gamma_5\psi^{(k)}(x):, \quad \text{pseudoscalar,}$$

where

(8.10a) $$\sigma_{\mu\nu} = (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/2i$$

is a Hermitian matrix corresponding to eq. (6.44b) and

(8.10b) $$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4.$$

The factors of i are included so that, when $j = k$, the operators are Hermitian (or anti-Hermitian when any one index $\mu = 4$).

That the expressions eqs. (8.9) transform under proper Lorentz transformations as indicated follows directly from the transformation properties of $\psi(x)$ given by eqs. (6.26) and (6.27). It remains to be determined how tensors having the form of these bilinear covariants transform under the kinematically admissible transformations **P**, **T**, and **C**. For this purpose, we replace the $\psi^{(j)}(x)$ in eqs. (8.9) by $\psi_0^{(j)}(x)$ and apply eqs. (7.39), (7.45) (with t^0 replaced by t), and eq. (7.67). The results for **P** are then found to be:

Action of P, C, T

(8.11a) $$\mathbf{P}:\bar{\psi}_0^{(j)}(x)\psi_0^{(k)}(x):\mathbf{P}^{-1} = :\bar{\psi}_0^{(j)}(\bar{x})\psi_0^{(k)}(\bar{x}):$$

(8.11b) $$\mathbf{P}:\bar{\psi}_0^{(j)}(x)\gamma_\mu\psi_0^{(k)}(x):\mathbf{P}^{-1} = -\varepsilon_\mu:\bar{\psi}_0^{(j)}(\bar{x})\gamma_\mu\psi_0^{(k)}(\bar{x}):$$

(8.11c) $$\mathbf{P}:\bar{\psi}_0^{(j)}(x)\sigma_{\mu\nu}\psi_0^{(k)}(x):\mathbf{P}^{-1} = \varepsilon_\mu\varepsilon_\nu:\bar{\psi}_0^{(j)}(\bar{x})\sigma_{\mu\nu}\psi_0^{(k)}(\bar{x}):$$

(8.11d) $$P{:}\bar{\psi}_0^{(j)}(x)\gamma_\mu\gamma_5\psi_0^{(k)}(x){:}P^{-1} = \varepsilon_\mu{:}\bar{\psi}_0^{(j)}(\bar{x})\gamma_\mu\gamma_5\psi_0^{(k)}(\bar{x}){:}$$

(8.11e) $$P{:}\bar{\psi}_0^{(j)}(x)\gamma_5\psi_0^{(k)}(x){:}P^{-1} = -{:}\bar{\psi}_0^{(j)}(\bar{x})\gamma_5\psi_0^{(k)}(\bar{x}){:},$$

where use has been made of the anticommutation of the gamma matrices. It should be noted that commutators with the generators Γ, Γ' of the unitary transformations are not affected by the normal ordering because the difference between the normal-ordered and ordinary product is a c-number ("classical number") function, not an operator.

The effect of T resulting from application of eqs. (7.85) and (6.40) is

(8.12a) $$T{:}\bar{\psi}_0^{(j)}(x)\psi_0^{(k)}(x){:}T^{-1} = {:}\bar{\psi}_0^{(j)}(x')\psi_0^{(k)}(x'){:}$$

(8.12b) $$Ti{:}\bar{\psi}_0^{(j)}(x)\gamma_\mu\psi_0^{(k)}(x){:}T^{-1} = -i{:}\bar{\psi}_0^{(j)}(x')\gamma_\mu\psi_0^{(k)}(x'){:}$$

(8.12c) $$T{:}\bar{\psi}_0^{(j)}(x)\sigma_{\mu\nu}\psi_0^{(k)}(x){:}T^{-1} = -{:}\bar{\psi}_0^{(j)}(x')\sigma_{\mu\nu}\psi_0^{(k)}(x'){:}$$

(8.12d) $$Ti{:}\bar{\psi}_0^{(j)}(x)\gamma_\mu\gamma_5\psi_0^{(k)}(x){:}T^{-1} = -i{:}\bar{\psi}_0^{(j)}(x')\gamma_\mu\gamma_5\psi_0^{(k)}(x'){:}$$

(8.12e) $$Ti{:}\bar{\psi}_0^{(j)}(x)\gamma_5\psi_0^{(k)}(x){:}T^{-1} = -i{:}\bar{\psi}_0^{(j)}(x')\gamma_5\psi_0^{(k)}(x'){:}.$$

From these transformations it can be seen that the bilinear covariants, eqs. (8.9), transform under T with the same phases as those assigned to the corresponding tensor fields in table 7.1, thereby establishing the correctness of that assignment as promised in connection with eq. (6.13b).[6]

Finally, the effect on those bilinear forms of charge conjugation is obtained from eq. (7.67) by use of eq. (7.66a), eq. (8.2), and

(8.13) $$\tilde{\gamma}_C = \gamma_C^{-1} = -\gamma_C,$$

where $\gamma_C = \gamma_2\gamma_4$:

(8.14a) $$C{:}\bar{\psi}_0^{(j)}(x)\psi_0^{(k)}(x){:}C^{-1} = {:}\bar{\psi}_0^{(k)}(x)\psi_0^{(j)}(x){:}$$

(8.14b) $$C{:}\bar{\psi}_0^{(j)}(x)\gamma_\mu\psi_0^{(k)}(x){:}C^{-1} = -{:}\bar{\psi}_0^{(k)}(x)\gamma_\mu\psi_0^{(j)}(x){:}$$

(8.14c) $$C{:}\bar{\psi}_0^{(j)}(x)\sigma_{\mu\nu}\psi_0^{(k)}(x){:}C^{-1} = -{:}\bar{\psi}_0^{(k)}(x)\sigma_{\mu\nu}\psi_0^{(j)}(x){:}$$

(8.14d) $$C{:}\bar{\psi}_0^{(j)}(x)\gamma_\mu\gamma_5\psi_0^{(k)}(x){:}C^{-1} = {:}\bar{\psi}_0^{(k)}(x)\gamma_\mu\gamma_5\psi_0^{(j)}(x){:}$$

(8.14e) $$C{:}\bar{\psi}_0^{(j)}(x)\gamma_5\psi_0^{(k)}(x){:}C^{-1} = {:}\bar{\psi}_0^{(k)}(x)\gamma_5\psi_0^{(j)}(x){:}.$$

The examples of interactions between tensor and spinor fields to be considered here are those Lagrangians that are obtained by construction of the products of these bilinear tensor operators with tensor fields of the same

[6] Note that although $\bar{\psi}\gamma_\mu\psi$ transforms like x_μ under proper Lorentz transformations, it is not possible to construct a Hermitian (anti-Hermitian for $\mu = 4$) operator transforming under T like x_μ, eq. (6.12a), because $(\bar{\psi}\gamma_\mu\psi)^\dagger = -\varepsilon_\mu\bar{\psi}\gamma_\mu\psi$.

TABLE 7.1 Transformed Forms of Tensor and Spinor Fields for Improper Transformations

Field	$\phi_0(x)$	$\phi_{\mu,0}(x)$	$\phi_{\mu\nu,0}(x)$	$\chi_{\mu,0}(x)$	$\chi_0(x)$[a]	$\psi_0(x)$
P:	$\phi_0(-\vec{x}, t)$	$-\varepsilon_\mu\phi_{\mu,0}(-\vec{x}, t)$	$\varepsilon_\mu\varepsilon_\nu\phi_{\mu\nu,0}(-\vec{x}, t)$	$\varepsilon_\mu\chi_{\mu,0}(-\vec{x}, t)$	$-\chi_0(-\vec{x}, t)$	$\pm\gamma_4\psi_0(-\vec{x}, t)$[b]
T:	$\phi_0(\vec{x}, -t)$	$-\phi_{\mu,0}(\vec{x}, -t)$	$-\phi_{\mu\nu,0}(\vec{x}, -t)$	$-\chi_{\mu,0}(\vec{x}, -t)$	$-\chi_0(\vec{x}, -t)$	$\sigma_2\psi_0(\vec{x}, -t)$
C:[c]	$\phi_0^\dagger(x)$	$-\varepsilon_\mu\phi_{\mu,0}^\dagger(x)$	$-\varepsilon_\mu\varepsilon_\nu\phi_{\mu\nu,0}^\dagger(x)$	$\varepsilon_\mu\chi_{\mu,0}^\dagger(x)$	$\chi_0^\dagger(x)$	$\gamma_2\bar{\psi}_0^\dagger(x)$

[a] In making reference to the text it should be noted that while ϕ has been used there to designate either the scalar field or the pesudoscalar field, the specific notation, χ, of section 6.1, case V, is used here for the pseudoscalar field to differentiate it clearly from the scalar field ϕ.

[b] The $\pm$ signs are associated with particle and antiparticle fields, respectively.

[c] The choice of the η_C associated with each type of tensor field is based on the convention that all spinor fields are assigned the same η_C. See note 9.

CPT Theorem

Lorentz covariant quantum field theory Lagrangians are invariant under combined action CPT

shown by brute force: Apply CPT to each term that can appear in Lagrangian

Invariance may fail for C, P, T, CP, CT, PT individually

Weak interaction violates P symmetry

⇓

CT also violated

⇓ if C respected

T violated

In field theory, Lagrangian inserted into theoretical machinery with no preferred C, P, T sense

All processes in quantum field theory are CPT invariant

(Allowed process ) $\xrightarrow{\text{Apply CPT}}$ (Allowed process )

PROBLEMS

- Brute complexity & opacity of the result.

Just why does it work?
How fragile is it?
What would it take to break it?

- Frank Arntzenius & Hilary Greaves:

How is it that restricted Lorentz covariance = invariance under changes of inertial states of motion

leads to

I CPT invariance

Charge? What has that to do with motion?

II Any other symmetry at all

First Problem Why C? : $CT_{\text{standard}} \equiv T_{\text{Arntzenius-Greaves}}$

so

$$\underset{\text{standard}}{\text{CPT Theorem}} \equiv \underset{\text{Arntzenius-Greaves}}{\text{PT Theorem}}$$

ie, "No "C""

Developed for Classical e-m in Frank Arntzenius & Hilary Greaves, "Time Reversal in Classical Electromagnetism"

Maxwell Electrodynamics is invariant under

$$\nabla \cdot E = \rho \qquad \nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \nabla \times B = \frac{\partial E}{\partial t} + j$$

$$F = q(E + v \times B)$$

(Extended to QFT in Arntzenius, "The CPT Theorem")

Standard Time Reversal T		Arntzenius-Greaves Time reversal T_{AG}
$-v \xleftarrow{T}$	v	$\xrightarrow{T_{AG}} -v$
$-j \longleftarrow$	j	$\longrightarrow j$
$E \longleftarrow$	E	$\longrightarrow -E$
$-B \longleftarrow$	B	$\longrightarrow B$
$\rho \longleftarrow$	ρ	$\longrightarrow -\rho$
$q \longleftarrow$	q	$\longrightarrow -q$
$\nabla \longleftarrow$	∇	$\longrightarrow \nabla$
$-t \longleftarrow$	t	$\longrightarrow -t$

] Hence "C" is built into T_{AG}

OOPS! SEE OVER

WARNING. April 10. 2009

The A-G transformations I wrote down were what I thought A-G had to intend. They correspond to the 4-D formalism transformation

$$q \longrightarrow -q$$
$$F_{ab} \longrightarrow F_{ab}$$
$$V^a \longrightarrow -V^a$$
$$j^a \longrightarrow j^a$$

From correspondence with Hilary Greaves & Frank A., it is now clear that they do not intend any. They really mean

$$q \rightarrow q$$
$$F_{ab} \rightarrow F_{ab}$$
$$\left.\begin{array}{l} V^a \rightarrow V^a \\ j^a \rightarrow j^a \end{array}\right]$$ so 3-velocity $\underline{v}$ is not flipped in sign by time reversal.

I am trying to sort this out in email. No success yet.

Possible T symmetries for Maxwell Electrodynamics

Not negotiable: $t \to -t$, $\underset{\sim}{v} \to -\underset{\sim}{v}$, $x,y,z \to x,y,z$, $\underset{\sim}{F} \to \underset{\sim}{F}$

Option ©: Allow parity to flip??

Negotiable: $\underbrace{q \to q}_{Ⓐ}$ or $\underbrace{q \to -q}_{Ⓑ}$

Option A

$q \to q$
$x,y,z \to x,y,z$

$\rho = d^3q/dx\,dy\,dz$

$\rho \to \rho$

$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \rho$
or $\underset{\sim}{F} = q\underset{\sim}{E}$ (for $\underline{B}=0$)

$\underset{\sim}{E} \to \underset{\sim}{E}$

$\underset{\sim}{j} = \rho\underset{\sim}{v}$

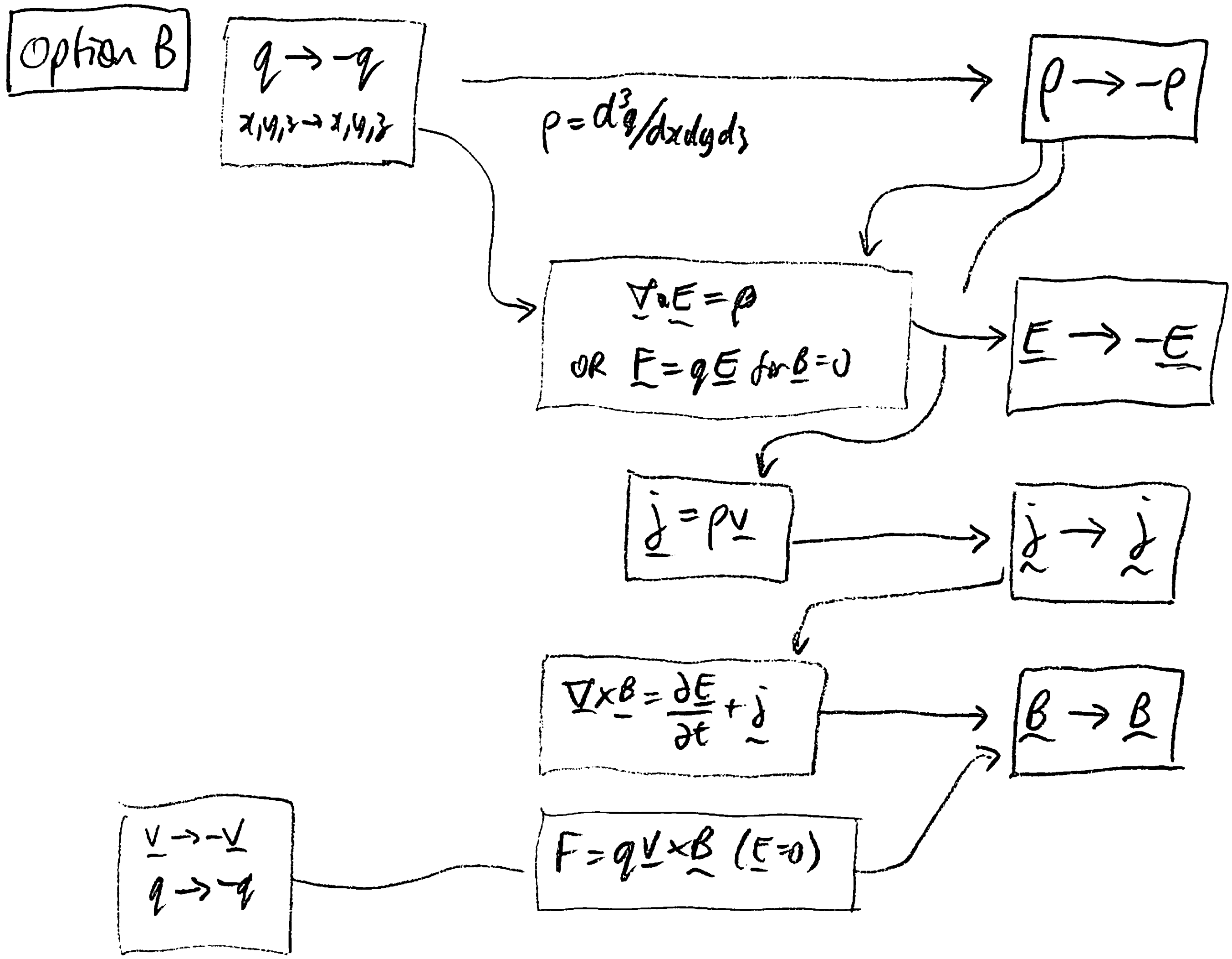
$\underset{\sim}{j} \to -\underset{\sim}{j}$

$\underset{\sim}{\nabla} \times \underline{B} = \dfrac{\partial \underline{E}}{\partial t} + \underset{\sim}{j}$

$\underline{B} \to -\underline{B}$

$\underline{v} \to -\underline{v}$
$q \to q$

$\underline{F} = q\underset{\sim}{v} \times \underline{B}$ ($\underline{E}=0$)



Options A and B each lead to unique results:

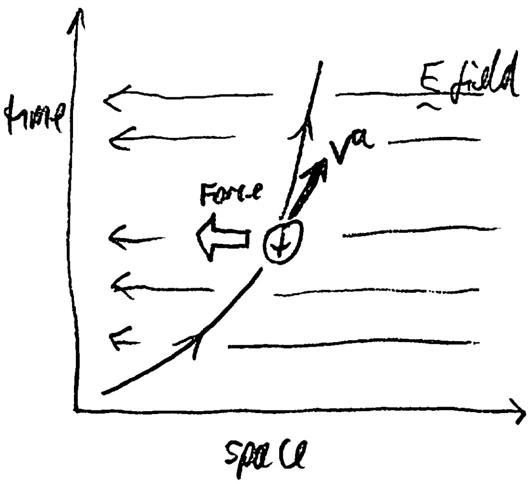
A → $T_{standard}$ B → T_{AG}

... same results from 4-D formalism

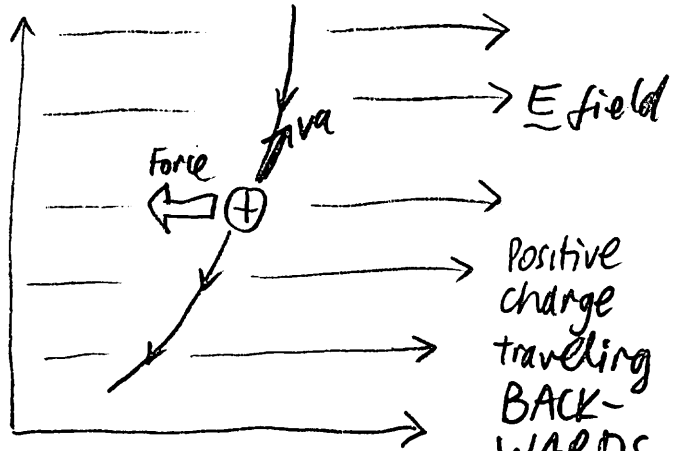
$F^a = q F^a{}_b V^b$ A T flips sign of $F^a{}_b, V^b$

B T flips sign of q, V^b

Feynman Rationale



time reverse ⇒



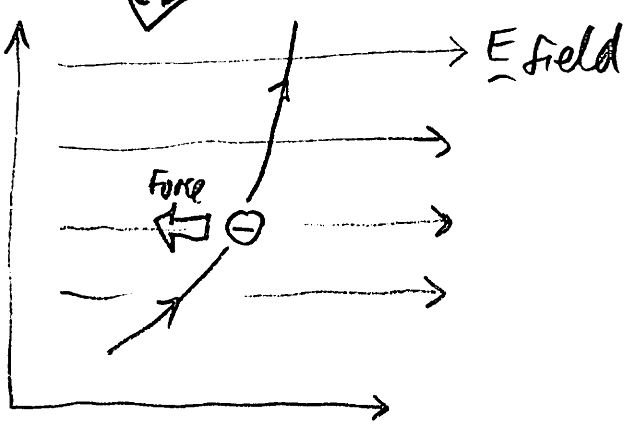
Positive charge traveling BACKWARDS in time

⇓

Reinterpret

⊕ charge traveling BACKWARDS in time ≡ ⊖ charge traveling FORWARDS in time

⇓



JDN
i'm not sure how this works. Somehow it assures us that ⊖ is just ⊕ going "backwards in time".

I am <u>more</u> assured by computing directly that T_{AG} is a symmetry of maxwell's theory

Which is the Right One?

AG: Time reversal transformation
p4. fixed by geometric structures of theory.

Different transformation ⇒ Different theory

p.16

$T_{standard}$ ↕ standard ontology (malament: no direction to trajectories)

T_{AG} ↕ Feynman ontology (world lines of charges are directed)

... but aren't both physically same theory?

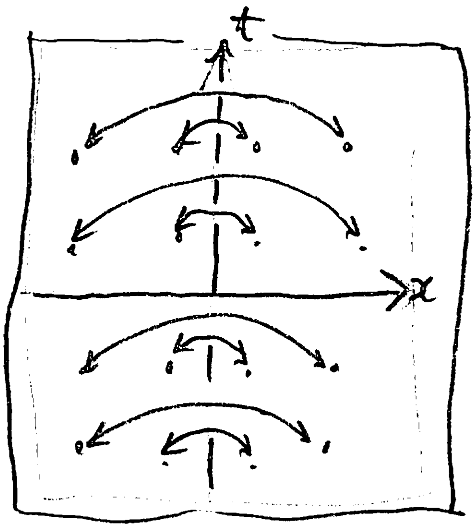
AG's resolution:
Difference between ontologies is only apparent, purely conventional.

JDN: This idea seems right, but the implementation is opaque to me

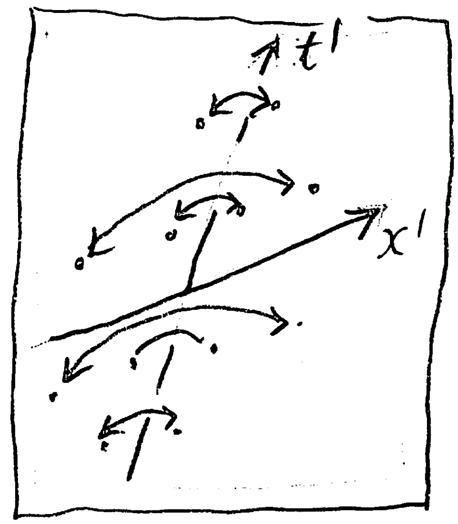
Norton's take
(may be same as A.G.)

Geometric structure does fix discrete symmetries ... but possibly not uniquely

Trivial example. Parity in 2-D minkowski spacetime



(x,t), (x',t') related by Lorentz transform.



P, P' are parity expressed in different frames

$P: (x,t) \longrightarrow (-x,t)$

$P': (x',t') \longrightarrow (-x',t')$

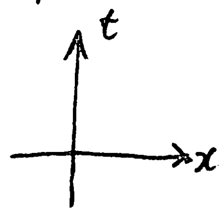
(Restricted) Lorentz transformation $L: \underset{\text{in 1st coord syst}}{(x,t)} \longrightarrow \underset{\text{in 2nd coord system}}{(x,t)}$ } i.e. active boost relates points with same coord. in the two coord. systems

P and P' are different transformations.
But they both express parity invariance of a minkowski spacetime, since they are connected by a transformation L which leaves parity undisturbed

$$P = L^{-1} P' L$$

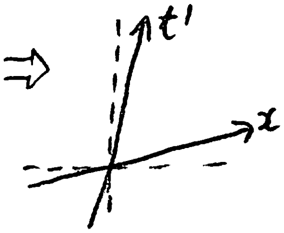
$$\boxed{P = L^{-1} P' L}$$

coordinate system S

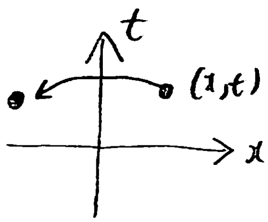


$$L \quad \begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(t - \tfrac{v}{c^2}x) \\ \gamma &= 1/\sqrt{1 - v^2/c^2} \end{aligned} \quad \Longrightarrow$$

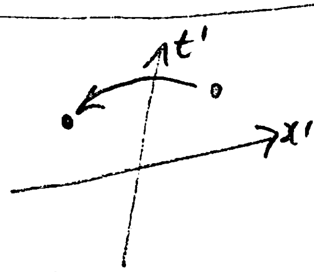
coordinate system S'



Define P, P', L as <u>active</u> boosts



$P: \underbrace{(x,t)}_{\text{in } S} \rightarrow (-x,t)$



$P': \underbrace{(x',t')}_{\text{in } S'} \rightarrow (-x',t')$

Active boost

$$L \quad \begin{aligned} x &\longrightarrow \bar{x} = \gamma(x + vt) \\ t &\longrightarrow \bar{t} = \gamma(t + \tfrac{v}{c^2}x) \end{aligned}$$

Key property

$L: \underbrace{\text{Point}}_{\text{coords in } S} \longrightarrow \underbrace{L\ \text{Point}}_{\text{coords in } S'}$

same

compute $L^{-1}PL$ for point (X,T) in S

$L: \underset{\text{in } S}{(X,T)} \xrightarrow{L} \underset{\text{in } S'}{(X,T)}$

P'

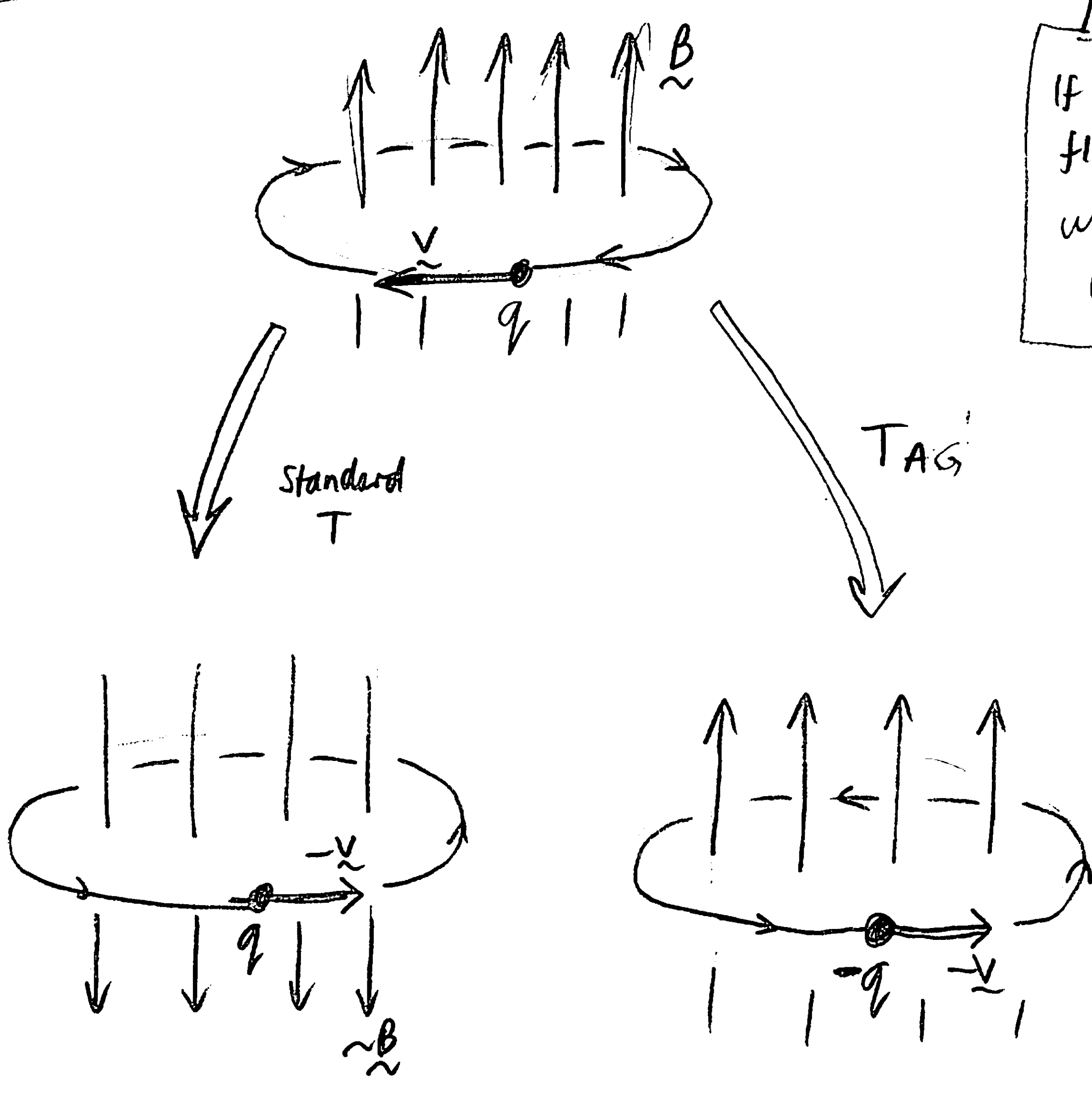
$P'L: \underset{\text{in } S}{(X,T)} \longrightarrow \underbrace{(-X,T)}_{\text{in } S'}$

L^{-1} via key property

$$\underbrace{L^{-1}P'L: \underbrace{(X,T)}_{\text{in } S} \longrightarrow \underbrace{(-X,T)}_{\text{in } S'}}_{\text{But this is just } P}$$

Electrodynamics admits two different time reversal symmetries

If it is OK to flip $\underset{\sim}{B}$ in T, why not flip q in T_{AG}?



degrees freedom associated with time

degrees freedom not associated with time

must reset something here to get T invariant.

Connected by charge conjugation

$q \longleftrightarrow -q$

A transformation unrelated to time reversal

Also symmetry of the theory

Charge conjugation symmetry of Maxwell theory

Two types of charges.
choose conventionally which is "positive"
This choice is encoded in
sign of $\underset{\sim}{E} \underset{\text{def}}{=} \underset{\sim}{F}/q$ ← switch "positive" & "negative" ⇓ switch sign on $\underset{\sim}{E}$

$$\underset{\sim}{\nabla}\cdot \underset{\sim}{E} = \rho \qquad \underset{\sim}{\nabla}\cdot \underset{\sim}{B} = 0$$

$$\underset{\sim}{\nabla}\times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} \qquad \underset{\sim}{\nabla}\times \underset{\sim}{B} = \frac{\partial \underset{\sim}{E}}{\partial t} + \underset{\sim}{j}$$

$$\underset{\sim}{v} \xrightarrow{C} \underset{\sim}{v}$$
$$\underset{\sim}{j} \xrightarrow{C} -\underset{\sim}{j}$$
$$\underset{\sim}{E} \xrightarrow{C} -\underset{\sim}{E}$$
$$\underset{\sim}{B} \xrightarrow{C} -\underset{\sim}{B}$$
$$\rho \xrightarrow{C} -\rho$$
$$q \xrightarrow{C} -q$$
$$\underset{\sim}{\nabla} \longrightarrow \underset{\sim}{\nabla}$$
$$t \longrightarrow t$$

compare earlier

$$T_{\text{standard}} = C\, T_{AG}$$

$$T_{AG} = C\, T_{\text{standard}}$$

Second Problem
Why should proper Lorentz symmetry extend at all?

The extension is due to which Lagrangians are admissible: NONE can pick out a preferred direction of time

Hilary Greaves, "Towards a geometrical understanding of the CPT Theorem"

Full Lorentz group divides into four isomorphic sectors

($L^\uparrow_+$) only here are transform. continuously connected to identity

$L^\uparrow_+$ Restricted Lorentz group

$L^\uparrow_-$

$L^\downarrow_-$

$L^\downarrow_+$

Proper Lorentz group

L_+

$\uparrow$ = Preserves time sense

$+$ = Determinant $+1$

($\therefore$ $-$ = reversed handedness of vierbein)

vier = 4

CLASSICAL THEOREM

If CPT is really just PT then $\therefore$ PT theorem says:

$$\boxed{\begin{array}{c}\text{Symmetry of theory is restricted Lorentz group } L^\uparrow_+ \\ \Downarrow \\ \text{Symmetry of theory is proper Lorentz group } L_+\end{array}}$$

L_+ contains

$$\underbrace{\text{PT: } \begin{array}{l} t \to -t \;\big]\, T \\ \left.\begin{array}{l} x \to -x \\ y \to -y \\ z \to -z \end{array}\right]\, P \end{array}}_{\text{Det } = +1}$$

Theorem explained:

Explicit restriction to restricted Lorentz group $L_+^\uparrow$ + Restriction to Lagrangians that cannot express the difference between $L_+^\uparrow$ and L_+ → Full theory that is invariant under L_+ (includes PT)

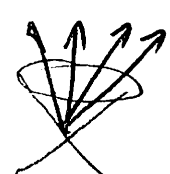
By conditions:

1. Dynamical fields are tensors
2. Dynamical equations are partial differential equations that are local polynomials in the fields & their derivatives.

P.22 Footnote

No tensor can pick out a direction of time : "any tensor invariant under $L_+^\uparrow$ is invariant under L_+"

Plausibility: Pick out time direction by:

-  ← selecting future lobe of lightcone? $\boxed{\text{Not a tensor}}$
- Pick out single time-like vector field  T^a $\boxed{\text{Too specific. } T^a \text{ is not invariant under } L_+^\uparrow}$
- Pick out all vector fields in future light cone  $\boxed{\text{Not introduced by differential equations}}$

Drop conditions 1., 2.,
& now theorem breaks

p.18

e.g. Base theory on $\underbrace{\text{pseudoscalar } \phi}_{\text{Flips sign under PT}}$

"Theory" $\phi = 1$ is not PT invariant

e.g. Theory with field equation

$$\underset{\text{scalar}}{\psi}\,\underset{\text{pseudoscalar}}{\chi} - \psi = 0$$

JBN
Presumably none of these are recoverable from a scalar Lagrangian

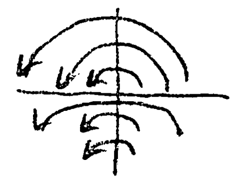
VERY TOY ILLUSTRATION
A P-theorem for 2D Euclidean geometry

Ordinary two dimensional Euclidean geometry is invariant under spatial rotations

$$\underbrace{R}_{\text{Full group of rotations}} = \underbrace{R_+}_{\text{Proper rotations}} + \underbrace{R_-}_{\text{Improper}}$$

Proper rotations: Det = 1

Improper = Rotation + Parity switch



Set up geometry as:

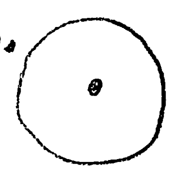
Requirement of proper rotational symmetry + Euclid's postulates = Geometry invariant under full rotation group

Euclid's postulates:

1. Always draw straight line between two points

⋮

3. circle of any center and radius



⋮

Euclid's postulates: cannot distinguish proper & improper rotation

Geometry invariant under full rotation group: Outcome of the "Parity Theorem"

Break theorem with:

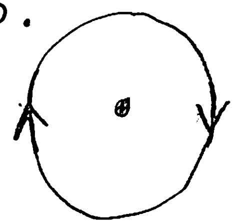
Requirement of proper rotational symmetry

+

Augmented Euclid's postulates

⋮

3*.



circle of any center & radius

circumference is DIRECTED line in CLOCKWISE sense

=

Geometry invariant under proper rotation only

Invariant under proper rotation only!