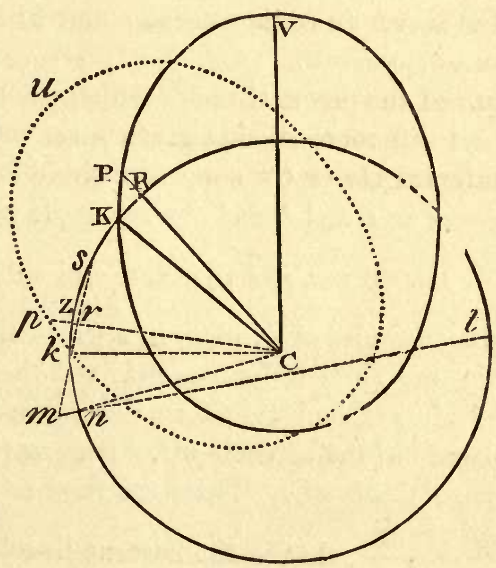
revolve with the point p in the curve line which the same point p, by the method just now explained, may be made to describe an immovable plane. Make the angle VCu equal to the angle PCp, and the line Cu equal to CV, and the figure uCp equal to the figure VCP, and the body being always in the point p, will move in the perimeter of the revolving figure uCp, and will describe its (revolving) arc up in the same time that the other body P describes the similar and equal arc VP in the quiescent figure VPK. Find, then, by Cor. 5, Prop. VI., the centripetal force by which the body may be made to revolve in the curve line which the point p describes in an immovable plane, and the Problem will be solved. Q.E.F.

PROPOSITION XLIV. THEOREM XIV.

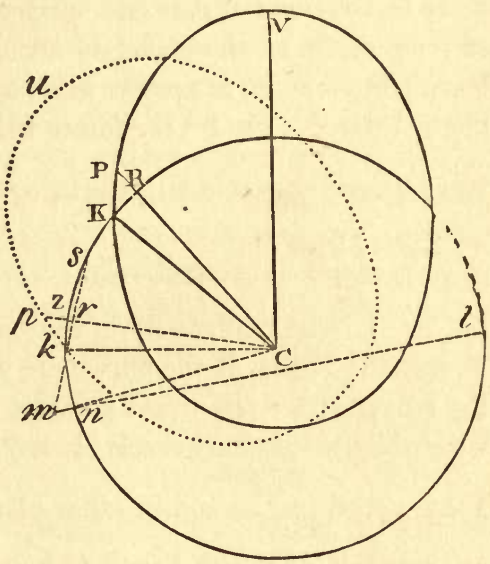
The difference of the forces, by which two bodies may be made to move equally, one in a quiescent, the other in the same orbit revolving, is in a triplicate ratio of their common altitudes inversely.



Let the parts of the quiescent orbit VP, PK be similar and equal to the parts of the revolving orbit up, pk; and let the distance of the points P and K be supposed of the utmost smallness Let fall a perpendicular kr from the point k to the right line pC, and produce it to m, so that mr may be to kr as the angle VCp to the angle VCP. Because the altitudes of the bodies PC and pC, KC and kC, are always equal, it is manifest that the increments or decrements of the lines PC and pC are always equal; and therefore if each of the several motions of the bodies in the places P and p be resolved into two (by Cor. 2 of the Laws of Motion), one of which is directed towards the centre, or according to the lines PC, pC, and the other, transverse to the former, hath a direction perpendicular to the lines PC and pC; the motions towards the centre will be equal, and the transverse motion of the body p will be to the transverse motion of the body P as the angular motion of the line pC to the angular motion of the line PC; that is, as the angle VCp to the angle VCP. Therefore, at the same time that the body P, by both its motions, comes to the point K, the body p, having an equal motion towards the centre, will be equally moved from p towards C; and therefore that time being expired, it will be found somewhere in the line mkr, which, passing through the point k, is perpendicular to the line pC; and by its transverse motion will acquire a distance from the line

vC, that will be to the distance which the other body P acquires from the line PC as the transverse motion of the body p to the transverse motion of the other body P. Therefore since kr is equal to the distance which the body P acquires from the line PC, and mr is to kr as the angle VCp to the angle VCP, that is, as the transverse motion of the body p to the transverse motion of the body P, it is manifest that the body p, at the expiration of that time, will be found in the place m. These things will be so, if the bodies p and P are equally moved in the directions of the lines pC and PC, and are therefore urged with equal forces in those directions. But if we take an angle pCn that is to the angle pCk as the angle VCp to the angle VCP, and nC be equal to kC, in that case the body p at the expiration of the time will really be in n; and is therefore urged with a greater force than the body P, if the angle nCp is greater than the angle kCp, that is, if the orbit upk, move either *in consequentia*, or *in antecedentia*, with a celerity greater than the double of that with which the line CP moves *in consequentia*; and with a less force if the orbit moves slower *in antecedentia*. And the difference of the forces will be as the interval mn of the places through which the body would be carried by the action of that difference in that given space of time. About the centre C with the interval Cn or Ck suppose a circle described cutting the lines mr, mn produced in s and t, and the rectangle $mn \times mt$ will be equal to the rectangle $mk \times ms$, and therefore mn will be equal to $\frac{mk \times ms}{mt}$. But since the triangles pCk, pCn, in a given time, are of a given magnitude, kr and mr, and their difference mk, and their sum ms, are reciprocally as the altitude pC, and therefore the rectangle $mk \times ms$ is reciprocally as the square of the altitude pC. But, moreover, mt is directly as $\frac{1}{2}mt$, that is, as the altitude pC. These are the first ratios of the nascent lines; and hence $\frac{mk \times ms}{mt}$, that is, the nascent lineola mn, and the difference of the forces proportional thereto, are reciprocally as the cube of the altitude pC. Q.E.D.

Cor. 1. Hence the difference of the forces in the places P and p, or K and k, is to the force with which a body may revolve with a circular motion from R to K, in the same time that the body P in an immovable orb describes the arc PK, as the nascent line mn to the versed sine of the nascent arc RK, that is, as $\frac{mk \times ms}{mt}$ to $\frac{rk^2}{2k\text{C}}$, or as $mk \times ms$ to the square of rk; that is, if we take given quantities F and G in the same ratio to one another as the angle VCP bears to the angle VCp, as GG — FF to FF. And, therefore, if from the centre C, with any distance CP or Cp, there be described a circular sector equal to the whole area VPC, which the body revolving in an immovable orbit has by a radius drawn to the centre described in any certain time, the difference of the forces, with which the body P revolves in an immovable orbit, and the body p in a movable orbit, will be to the centripetal force, with which another body by a radius drawn to the centre can uniformly describe that sector in the same time as the area VPC is described, as GG — FF to FF. For that sector and the area pCk are to one another as the times in which they are described.



COR. 2. If the orbit VPK be an ellipsis, having its focus C, and its highest apsis V, and we suppose the the ellipsis upk similar and equal to it, so that pC may be always equal to PC, and the angle VCp be to the angle VCP in the given ratio of G to F; and for the altitude PC or pC we put A, and 2R for the latus rectum of the ellipsis, the force with which a body may be made to revolve in a movable ellipsis will be as $\frac{FF}{AA} + \frac{RGG - RFF}{A^3}$, and *vice versa*. Let the force with which a body may revolve in an immovable ellipsis be expressed by the quantity $\frac{FF}{AA}$, and the force in V will be $\frac{FF}{CV^2}$. But the force with which a body may revolve in a circle at the distance CV, with the same velocity as a body revolving in an ellipsis has in V, is to the force with which a body revolving in an ellipsis is acted upon in the apsis V, as half the latus rectum of the ellipsis to the semi-diameter CV of the circle, and therefore is as $\frac{RFF}{CV^3}$; and the force which is to this, as GG — FF to FF, is as $\frac{RGG - RFF}{CV^3}$: and this force (by Cor. 1 of this Prop.) is the difference of the forces in V, with which the body P revolves in the immovable ellipsis VPK, and the body p in the movable ellipsis upk. Therefore since by this Prop. that difference at any other altitude A is to itself at the altitude CV as $\frac{1}{A^3}$ to $\frac{1}{CV^3}$, the same difference in every altitude A will be as $\frac{RGG - RFF}{A^3}$. Therefore to the force $\frac{FF}{AA}$, by which the body may revolve in an immovable ellipsis VPK

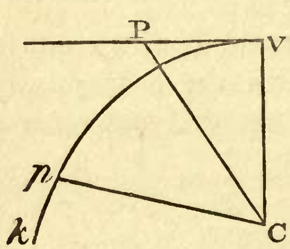
add the excess $\frac{RGG-RFF}{A^3}$, and the sum will be the whole force $\frac{FF}{AA}+\frac{RGG-RFF}{A^3}$ by which a body may revolve in the same time in the movable ellipsis *upk*.

Cor. 3. In the same manner it will be found, that, if the immovable orbit VPK be an ellipsis having its centre in the centre of the forces C, and there be supposed a movable ellipsis *upk*, similar, equal, and concentrical to it; and 2R be the principal latus rectum of that ellipsis, and 2T the latus transversum, or greater axis; and the angle VC*p* be continually to the angle VCP as G to F; the forces with which bodies may revolve in the immovable and movable ellipsis, in equal times, will be as $\frac{FFA}{T^3}$ and $\frac{FFA}{T^3}+\frac{RGG-RFF}{A^3}$ respectively.

Cor. 4. And universally, if the greatest altitude CV of the body be called T, and the radius of the curvature which the orbit VPK has in V, that is, the radius of a circle equally curve, be called R, and the centripetal force with which a body may revolve in any immovable trajectory VPK at the place V be called $\frac{VFF}{TT}$, and in other places P be indefinitely styled X; and the altitude CP be called A, and G be taken to F in the given ratio of the angle VC*p* to the angle VCP; the centripetal force with which the same body will perform the same motions in the same time, in the same trajectory *upk* revolving with a circular motion, will be as the sum of the forces $X+\frac{VRGG-VRFF}{A^3}$.

Cor. 5. Therefore the motion of a body in an immovable orbit being given, its angular motion round the centre of the forces may be increased or diminished in a given ratio; and thence new immovable orbits may be found in which bodies may revolve with new centripetal forces.

Cor. 6. Therefore if there be erected the line VP of an indeterminate length, perpendicular to the line CV given by position, and CP be drawn, and C*p* equal to it, making the angle VC*p* having a given ratio to the angle VCP, the force with which a body may revolve in the curve line V*pk*, which the point *p* is continually describing, will be reciprocally as the cube of the altitude C*p*. For the body P, by its *vis inertiæ* alone, no other force impelling it, will proceed uniformly in the right line VP. Add, then, a force tending to the centre C reciprocally as the cube of the altitude CP or C*p*, and (by what was just demonstrated) the



body will deflect from the rectilinear motion into the curve line V*pk*. But this curve V*pk* is the same with the curve VPQ found in Cor. 3, Prop XLI, in which, I said, bodies attracted with such forces would ascend obliquely.

PROPOSITION XLV. PROBLEM XXXI.

To find the motion of the apsides in orbits approaching very near to circles.

This problem is solved arithmetically by reducing the orbit, which a body revolving in a movable ellipsis (as in Cor. 2 and 3 of the above Prop.) describes in an immovable plane, to the figure of the orbit whose apsides are required; and then seeking the apsides of the orbit which that body describes in an immovable plane. But orbits acquire the same figure, if the centripetal forces with which they are described, compared between themselves, are made proportional at equal altitudes. Let the point V be the highest apsis, and write T for the greatest altitude CV, A for any other altitude CP or C*p*, and X for the difference of the altitudes CV — CP; and the force with which a body moves in an ellipsis revolving about its focus C (as in Cor. 2), and which in Cor. 2 was as $\frac{FF}{AA} + \frac{RGG - RFF}{A^3}$, that is as, $\frac{FFA + RGG - RFF}{A^3}$, by substituting T — X for A, will become as $\frac{RGG - RFF + TFF - FFX}{A^3}$. In like manner any other centripetal force is to be reduced to a fraction whose denominator is A^3, and the numerators are to be made analogous by collating together the homologous terms. This will be made plainer by Examples.

EXAMPLE 1. Let us suppose the centripetal force to be uniform, and therefore as $\frac{A^3}{A^3}$ or, writing T — X for A in the numerator, as $\frac{T^3 - 3TTX + 3TXX - X^3}{A^3}$. Then collating together the correspondent terms of the numerators, that is, those that consist of given quantities, with those of given quantities, and those of quantities not given with those of quantities not given, it will become RGG — RFF + TFF to T^3 as — FFX to $3TTX + 3TXX - X^3$, or as —FF to —3TT + 3TX — XX. Now since the orbit is supposed extremely near to a circle, let it coincide with a circle; and because in that case R and T become equal, and X is infinitely diminished, the last ratios will be, as RGG to T^2, so — FF to — 3TT, or as GG to TT, so FF to 3TT; and again, as GG to FF, so TT to 3TT, that is, as 1 to 3; and therefore G is to F, that is, the angle VC*p* to the angle VCP, as 1 to $\sqrt{3}$. Therefore since the body, in an immovable ellipsis, in descending from the upper to the lower apsis, describes an angle, if I may so speak, of 180 deg., the other body in a movable ellipsis, and therefore in the immovable orbit we are treating of, will in its descent from the upper to the lower apsis, describe an angle VC*p* of $\frac{180}{\sqrt{3}}$ deg. And this comes to pass by reason of the likeness of this orbit which a body acted upon by an uniform centripetal force describes, and of that orbit which a body performing its circuits in a revolving ellipsis will describe in a quiescent plane. By this collation of the terms, these orbits are made similar; not universally, indeed, but then only when they approach very near to a circular figure. A body, therefore revolving with an uniform centripetal force in an orbit nearly circular, will always describe an angle of $\frac{180}{\sqrt{3}}$ deg., or 103 deg., 55 m., 23 sec., at the centre; moving from the upper apsis to the lower apsis when it has once described that angle, and thence returning to the upper apsis when it has described that angle again; and so on *in infinitum.*

Exam. 2. Suppose the centripetal force to be as any power of the altitude A, as, for example, A^{n-3}, or $\frac{A^n}{A^3}$; where $n - 3$ and n signify any indices of powers whatever, whether integers or fractions, rational or surd, affirmative or negative. That numerator A^n or $\overline{T - X}|^n$ being reduced to an indeterminate series by my method of converging series, will become $T^n - nXT^{n-1} + \frac{nn - n}{2} XXT^{n-2}$, &c. And conferring these terms with the terms of the other numerator RGG — RFF + TFF — FFX, it becomes as RGG — RFF + TFF to T^n, so — FF to $- nT^{n-1} + \frac{nn - n}{2} XT^{n-2}$, &c. And taking the last ratios where the orbits approach to circles, it becomes as RGG to T^n, so — FF to $- nT^{n-1}$, or as GG to T^{n-1}, so FF to nT^{n-} ; and again, GG to FF, so T^{n-1} to nT^{n-1}, that is, as 1 to *n*; and therefore G is to F, that is the angle VC*p* to the angle VCP, as 1 to $\sqrt{n}$. Therefore since the angle VCP, described in the descent of the body from the upper apsis to the lower apsis in an ellipsis, is of 180 deg., the angle VC*p*, described in the descent of the body from the upper apsis to the lower apsis in an orbit nearly circular which a body describes with a centripetal force proportional to the power A^{n-3}, will be equal to an angle of $\frac{180}{\sqrt{n}}$ deg., and this angle being repeated, the body will return from the lower to the upper apsis, and so on *in infinitum.* As if the centripetal force be as the distance of the body from the centre, that is, as A, or $\frac{A^4}{A^3}$, n will be equal to 4, and $\sqrt{n}$ equal to 2; and therefore the angle between the upper and the lower apsis will be equal to $\frac{180}{2}$ deg., or 90 deg. Therefore the body having performed a fourth part of one revolution, will arrive at the lower apsis, and having performed another fourth part, will arrive at the upper apsis, and so on by turns *in infinitum.* This appears also from Prop. X. For a body acted on by this centripetal force will revolve in an immovable ellipsis, whose centre is the centre of force. If the centripetal force is reciprocally as the distance, that is, directly as $\frac{1}{A}$ or $\frac{A^2}{A^3}$, n will be equal to 2; and therefore the angle between the upper and lower apsis will be $\frac{180}{\sqrt{2}}$ deg., or 127 deg., 16 min., 45 sec.; and therefore a body revolving with such a force, will by a perpetual repetition of this angle, move alternately from the upper to the lower and from the lower to the upper apsis for ever. So, also, if the centripetal force be reciprocally as the biquadrate root of the eleventh power of the altitude, that is, reciprocally as $A^{\frac{11}{4}}$, and, therefore, directly as $\frac{1}{A^{\frac{11}{4}}}$, or as $\frac{A^{\frac{1}{4}}}{A^3}$, n will be equal to $\frac{1}{4}$, and $\frac{180}{\sqrt{n}}$ deg. will be equal to 360 deg.; and therefore the body parting from the upper apsis, and from thence perpetually descending, will arrive at the lower apsis when it has completed one entire revolution; and thence ascending perpetually, when it has completed another entire revolution, it will arrive again at the upper apsis; and so alternately for ever.

EXAM. 3. Taking m and n for any indices of the powers of the altitude, and b and c for any given numbers, suppose the centripetal force to be as $\frac{bA^m + cA^n}{A^3}$, that is, as $\frac{b \text{ into } \overline{T - X}|^m + c \text{ into } \overline{T - X}|^n}{A^3}$ or (by the method of converging series above-mentioned) as

$$\frac{bT^m + cT^n - mbXT^{m-1} \quad ncXT^{n-1} + \frac{mm - m}{2} bXXT^{m-2} + \frac{nn - n}{2} cXXT^{n-2}, \&c.}{A^3}$$

and comparing the terms of the numerators, there will arise RGG — RFF + TFF to $bT^m + cT^n$ as — FF to $-mbT^{m-1} - ncT^{n-1} + \frac{mm - m}{2} bXT^{m-2} + \frac{nn - n}{2} cXT^{n-2}$, &c. And taking the last ratios that arise when the orbits come to a circular form, there will come forth GG to $bT^{m-1} + cT^{n-1}$ as FF to $mbT^{m-1} + ncT^{n-1}$; and again, GG to FF as $bT^{m-1} + cT^{n-1}$ to $mbT^{n-1} + ncT^{n-1}$. This proportion, by expressing the greatest altitude CV or T arithmetically by unity, becomes, GG to FF as $b + c$ to $mb + nc$, and therefore as 1 to $\frac{mb + nc}{b + c}$. Whence G becomes to F, that is, the angle VCp to the angle VCP, as 1 to $\sqrt{\frac{mb + nc}{b + c}}$. And therefore since the angle VCP between the upper and the lower apsis, in an immovable ellipsis, is of 180 deg., the angle VCp between the same apsides in an orbit which a body describes with a centripetal force, that is, as $\frac{bA^m + cA^n}{A^3}$, will be equal to an angle of $180\sqrt{\frac{b + c}{mb + nc}}$ deg. And by the same reasoning, if the centripetal force be as $\frac{bA^m - cA^n}{A^3}$, the angle between the apsides will be found equal to $180\sqrt{\frac{b - c}{mb - nc}}$ deg. After the same manner the Problem is solved in more difficult cases. The quantity to which the centripetal force is proportional must always be resolved into a converging series whose denominator is A^3. Then the given part of the numerator arising from that operation is to be supposed in the same ratio to that part of it which is not given, as the given part of this numerator RGG — RFF + TFF — FFX is to that part of the same numerator which is not given. And taking away the superfluous quantities, and writing unity for T, the proportion of G to F is obtained.

Cor. 1. Hence if the centripetal force be as any power of the altitude, that power may be found from the motion of the apsides; and so contrariwise. That is, if the whole angular motion, with which the body returns to the same apsis, be to the angular motion of one revolution, or 360 deg., as any number as m to another as n, and the altitude called A; the force will be as the power $A^{\frac{nn}{mm} - 3}$ of the altitude A; the index of which power is $\frac{nn}{mm} - 3$. This appears by the second example. Hence it is plain that the force in its recess from the centre cannot decrease in a greater than a triplicate ratio of the altitude. A body revolving with such a force, and parting from the apsis, if it once begins to descend, can never arrive at the lower apsis or least altitude, but will descend to the centre, describing the curve line treated of in Cor. 3, Prop. XLI. But if it should, at its parting from the lower apsis, begin to ascend never so little, it will ascend *in infinitum*, and never come to the upper apsis; but will describe the curve line spoken of in the same Cor., and Cor. 6, Prop. XLIV. So that where the force in its recess from the centre decreases in a greater than a triplicate ratio of the altitude, the body at its parting from the apsis, will either descend to the centre, or ascend *in infinitum*, according as it descends or ascends at the beginning of its motion. But if the force in its recess from the centre either decreases in a less than a triplicate ratio of the altitude, or increases in any ratio of the altitude whatsoever, the body will never descend to the centre, but will at some time arrive at the lower apsis; and, on the contrary, if the body alternately ascending and descending from one apsis to another never comes to the centre, then either the force increases in the recess from the centre, or it decreases in a less than a triplicate ratio of the altitude; and the sooner the body returns from one apsis to another, the farther is the ratio of the forces from the triplicate ratio. As if the body should return to and from the upper apsis by an alternate descent and ascent in 8 revolutions, or in 4, or 2, or $1\frac{1}{2}$; that is, if m should be to n as 8, or 4, or 2, or $1\frac{1}{2}$ to 1, and therefore $\frac{nn}{mm} - 3$, be $\frac{1}{64} - 3$, or $\frac{1}{16} - 3$, or $\frac{1}{4} - 3$, or $\frac{4}{9} - 3$; then the force will be as $A^{\frac{1}{64} - 3}$, or $A^{\frac{1}{16} - 3}$, or $A^{\frac{1}{4} - 3}$, or $A^{\frac{4}{9} - 3}$; that is, it will be reciprocally as $A^{3 - \frac{1}{64}}$, or $A^{3 - \frac{1}{16}}$, or $A^{3 - \frac{1}{4}}$, or $A^{3 - \frac{4}{9}}$. If the body after each revolution returns to the same apsis, and the apsis remains unmoved, then m will be to n as 1 to 1, and therefore $A^{\frac{nn}{mm} - 3}$ will be equal to A^{-2}, or $\frac{1}{AA}$; and therefore the decrease of the forces will be in a duplicate ratio of the altitude; as was demonstrated above. If the body in three fourth parts, or two thirds, or one third, or one fourth part of an entire revolution, return to the same apsis; m will be to n as $\frac{3}{4}$ or $\frac{2}{3}$ or $\frac{1}{3}$ or $\frac{1}{4}$ to 1, and therefore $A^{\frac{nn}{mm} - 3}$ is equal to $A^{\frac{16}{9} - 3}$, or $A^{\frac{9}{4} - 3}$, or $A^{9 - 3}$, or $A^{16 - 3}$; and therefore the force is either reciprocally as $A^{\frac{11}{9}}$ or $A^{\frac{3}{4}}$, or directly as A^{6} or A^{13}. Lastly if the body in its progress from the upper apsis to the same upper apsis again, goes over one entire revolution and three deg. more, and therefore that apsis in each revolution of the body moves three deg. *in consequentia;* then m will be to n as 363 deg. to 360 deg. or as 121 to 120, and therefore $A^{\frac{nn}{mm} - 3}$ will be equal to $A^{-\frac{29523}{14641}}$, and therefore the centripetal force will be reciprocally as $A^{\frac{29523}{14641}}$, or reciprocally as $A^{2\frac{4}{243}}$ very nearly. Therefore the centripetal force decreases in a ratio something greater than the duplicate; but approaching $59\frac{3}{4}$ times nearer to the duplicate than the triplicate.

COR. 2. Hence also if a body, urged by a centripetal force which is reciprocally as the square of the altitude, revolves in an ellipsis whose focus is in the centre of the forces; and a new and foreign force should be added to or subducted from this centripetal force, the motion of the apsides arising from that foreign force may (by the third Example) be known; and so on the contrary. As if the force with which the body revolves in the ellipsis be as $\frac{1}{AA}$; and the foreign force subducted as cA, and therefore the remaining force as $\frac{A - cA^4}{A^3}$; then (by the third Example) b will be equal to 1, m equal to 1, and n equal to 4; and therefore the angle of revolution between the apsides is equal to $180\sqrt{\frac{1-c}{1-4c}}$ deg. Suppose that foreign force to be 357.45 parts less than the other force with which the body revolves in the ellipsis; that is, c to be $\frac{100}{35745}$; A or T being equal to 1; and then $180\sqrt{\frac{1-c}{1-4c}}$ will be $180\sqrt{\frac{35645}{35345}}$ or 180.7623, that is, 180 deg., 45 min., 44 sec. Therefore the body, parting from the upper apsis, will arrive at the lower apsis with an angular motion of 180 deg., 45 min., 44 sec, and this angular motion being repeated, will return to the upper apsis; and therefore the upper apsis in each revolution will go forward 1 deg., 31 min., 28 sec. The apsis of the moon is about twice as swift

So much for the motion of bodies in orbits whose planes pass through the centre of force. It now remains to determine those motions in eccentrical planes. For those authors who treat of the motion of heavy bodies used to consider the ascent and descent of such bodies, not only in a perpendicular direction, but at all degrees of obliquity upon any given planes; and for the same reason we are to consider in this place the motions of bodies tending to centres by means of any forces whatsoever, when those bodies move in eccentrical planes. These planes are supposed to be perfectly smooth and polished, so as not to retard the motion of the bodies in the least. Moreover, in these demonstrations, instead of the planes upon which those bodies roll or slide, and which are therefore tangent planes to the bodies, I shall use planes parallel to them, in which the centres of the bodies move, and by that motion describe orbits. And by the same method I afterwards determine the motions of bodies performed in curve superficies.

SECTION X.

Of the motion of bodies in given superficies, and of the reciprocal motion of funependulous bodies.

PROPOSITION XLVI. PROBLEM XXXII.

Any kind of centripetal force being supposed, and the centre of force, and any plane whatsoever in which the body revolves, being given, and the quadratures of curvilinear figures being allowed; it is required to determine the motion of a body going off from a given place, with a given velocity, in the direction of a given right line in that plane.